MATHEMATICS MAGAZINE

Vol. 25, No. 3, Jan.-Feb., 1952

CONTENTS

OUR CONTRIBUTORS

Edward Kasner, Professor Emeritus of Mathematics, Columbia University, was born in New York in 1878. After graduating from C.C.N.Y. (B.S. '96) and Columbia (Ph.D. '99) he spent a year at Gottingen, and then returned to join the staff of Columbia. A member of the National Academy of Sciences, Prof. Kasner has also been honored by election to Vice-Presidency of the Am. Math. Soc. and of section A of the A.A.A.S. He is well known by mathematicians for his many contributions to differential geometry, dynamics, polygenic functions and other fields and by others as co-author of "Mathematics and the Imagination".

John J. DeCicco, Professor of Mathematics, De Paul University, attended Brooklyn College (B.S. '33) and Columbia (Ph.D. '38). Before being appointed to his present position, he was an Associate Professor at the Illinois Institute of Technology. A specialist in geometry and dynamics, Professor DeCicco often collaborates with Dr. Kasner, as in the present instance.

Curtis M. Fulton, Assistant Professor of Mathematics, University of California, Davis, attended the University and the Technische Hochschule in Munich (Ph.D. '36). After teaching several years at the Escuela Normal Superior, Bogota, Colombia, he came to California and has been on the Davis faculty since 1947.

Roger Cook Osborn, Instructor in Applied Mathematics, University of Texas, was born in Texas in 1920. An alumnus of the University of Texas (B.A. '40, M.A. '42), he taught in the Texas public schools before taking his present position in 1942. Since then he has also served for two years as a naval officer.

Pedro A. Piza was born in Arecibo, Puerto Rico in 1896. He attended the University of Michigan and the Massachusetts Institute of Technology and returned to Puerto Rico in 1919, where he has been in business ever since. An amateur mathematician, largely self-taught, his special interests are number theory and geometry. Besides articles in various periodicals, Mr. Piza has published a book entitled "Fermagoric Triangles".

THE OSCULATING CONICS OF PHYSICAL SYSTEMS OF CURVES

Edward Kasner and John DeCicco

1. *Introduction.* Kasner has presented the development of the osculating parabolas (four point contact) of the trajectories and the systems S_k of a positional field of force in the plane in the Princeton Colloquium[1]. In the present paper, we shall begin the study of the theory of the osculating conic sections (five point contact) of trajectories and general physical systems of curves.

2. *Systems S_k.* Consider a general positional field of force defined over a certain region of the (x, y)-plane where the force vector, acting at any point (x, y), is assumed to be continuous and to possess continuous partial derivatives of the first and second orders. Also the force vector is assumed to be not identically zero. There is no loss in generality in supposing that a particle moving in this field of force is of unit mass.

A *system S_k* of curves in this positional field of force consists of curves along which a constrained motion is possible so that the pressure P is proportional to the normal component N of the force vector. Thus $P = kN$ where k is the constant factor ($\neq -1$) of proportionality[2].

Let θ and r denote the inclination to the x-axis and the radius of curvature of a curve C, and let the subscript s denote total differentiation with respect to its arc length s. Also let t denote the time, and v the speed of a particle describing this curve C. Finally T and N denote the tangential and normal components of the force vector along a curve C in the given field of force.

The intrinsic differential equations of a system S_k are

$$(1) \qquad \frac{v^2}{r} = (k+1)N \qquad \frac{dv}{dt} = vv_s = T,$$

since $P = \dfrac{v^2}{r} - N = kN$. Eliminating the speed v from these equations, it is found that *the intrinsic differential equation of third order representing a system S_k in a positional field of force is*

$$(2) \qquad r_s = n\frac{T}{n} - r\frac{N_s}{N}, \quad \textit{where } n = \frac{2}{k+1}$$

The important systems S_k of physical interest are

(a) The system S_0 of trajectories given by $k = 0$, or $n = 2$.

(*b*) The system S_1 of general catenaries given by $k = 1$, or $n = 1$.

(*c*) The system S_{-2} of generalized brachistochrones given by $k = -2$, or $n = -2$.

(*d*) The system S_∞ of velocity curves given by $k = \infty$, or $n = 0$.

3. *Certain related vectors associated with a positional field of force.* We shall consider certain vectors derived from the force vector in order that the derivatives of higher order of (2) with respect to the arc length be of a simpler form.

Let $\phi(x, y)$ and $\psi(x, y)$ denote the horizontal and vertical components of the force vector. Then

$$T = \phi \cos \theta + \psi \sin \theta, \quad N = -\phi \sin \theta + \psi \cos \theta. \tag{3}$$

It is observed that T and N are any two differentiable functions of (x,y,θ) which obey the two relations

$$T_\theta = N, \quad N_\theta = -T. \tag{4}$$

Our two new vectors are the following ones. The horizontal and vertical components of the first vector are N_x and N_y, and those of the second vector are T_x and T_y. Denote the tangential and normal components of the first vector by A and B, and those of the second vector by C and D. Then

$$\begin{aligned} A &= N_x\cos \theta + N_y\sin \theta, \quad B = -N_x\sin \theta + N_y\cos \theta, \\ C &= T_x\cos \theta + T_y\sin \theta, \quad D = -T_x\sin \theta + T_y\cos \theta. \end{aligned} \tag{5}$$

Let E denote the quantity

$$E = A_x\cos \theta + A_y\sin \theta = N_{xx}\cos^2\theta + 2N_{xy}\cos \theta \sin \theta + N_{yy}\sin^2\theta. \tag{6}$$

From equations (2), (4), (5), (6), we obtain

$$\begin{aligned} Nr_s &= (n + 1)T - Ar, \\ N^2rr_{ss} &= (n + 1)(N^2 + T^2) + r[n(CN - 2AT) + \\ &\quad (2CN - 3AT - BN)] + r^2(2A^2 - EN). \end{aligned} \tag{7}$$

The following expressions are useful for later purposes.

$$\begin{aligned} N^2(9 + r_s^2 - 3rr_{ss}) &= (n - 2)[(n + 1)T^2 - 3N^2] \\ &+ r[n(4AT - 3CN) + (7AT - 6CN + 3BN)] + r^2(3EN - 5A^2), \\ N^2(9 + 2r_s^2 - 3rr_{ss}) &= (n + 1)(2n - 1)T^2 - 3(n - 2)N^2 \\ &+ r[n(2AT - 3CN) + (5AT - 6CN + 3BN)] + r^2(3EN - 4A^2). \end{aligned} \tag{8}$$

4. *The osculating conic sections of a system S_k of ∞^3 curves.* Consider a fixed lineal-element E defined by the point (x_0, y_0) and the direction through this point of inclination θ to the positive x-axis. Denote by (X, Y) the new cartesian coordinates of a point where the origin of this new coordinate system is at the point of E and the positive X-axis is the positive direction of E. The cartesian coordinates (X, Y) are said to be *relative* to the lineal-element E. The relationships between the old coordinates (x, y) and the new coordinates (X, Y) are

$$(x + iy) = e^{i\theta}(X + iY) + (x_0 + iy_0), \tag{9}$$

$$X + iY = [(x - x_0) + i(y - y_0)]e^{-i\theta}.$$

Next consider a curve $x = x(s)$. $y = y(s)$, where s is the arc length and r is the radius of curvature. Let $s = s_0$ define a fixed lineal-element E of the curve with point $x_0 = x(s_0)$, $y_0 = y(s_0)$, and direction $\theta = \arctan y_s(s_0)/x_s(s_0)$. The equation of the osculating conic section (five-point contact)[3] of the curve $x = x(s)$, $y = y(s)$, at the lineal-element E is

$$9X^2 - 6r_s XY + (9 + 2r_s{}^2 - 3rr_{ss})Y^2 - 18rY = 0, \tag{10}$$

where (X, Y) are the running coordinates of a point on the conic relative to the lineal-element E.

The conic section (10) is an ellipse, parabola, hyperbola according as the expression

$$9 + r_s{}^2 - 3rr_{ss}, \tag{11}$$

is positive, zero, negative.

If (11) is not zero, the conic section (10) is central. In this case, its center is

$$Z = X + iY = \frac{3r(r_s + 3i)}{9 + r_s{}^2 - 3rr_{ss}}. \tag{12}$$

The foci of the conic section (10) are

$$(9 + r_s{}^2 - 3rr_{ss})Z^2 - 6r(r_s + 3i)Z - 9r^2 = 0. \tag{13}$$

By substituting (7) and (8) into (10), it is found that *the osculating conic sections of a system S_k of ∞^3 curve in a positional field of force are*

$$\begin{aligned} &9N^2X^2 - 6N[(n + 1)T - Ar]XY \\ &+ [\{(n + 1)(2n - 1)T^2 - 3(n - 2)N^2\} + \{n(2AT - 3CN) \\ &+ (5AT - 6CN + 3BN)\}r + (3EN - 4A^2)r^2]Y^2 - 18N^2rY = 0. \end{aligned} \tag{14}$$

By (8) and (11), it is deduced that *there is, in general, one trajectory of the system* S_0 *of dynamical trajectories through a given lineal-element* E *which is hyperosculated by its osculating parabola. For* $k \neq 0$, *there are, in general, two curves of a system* S_k *of curves through a given lineal-element* E *which are hyperosculated by their osculating parabolas.*

The family (14) is quadratic in r and also quadratic in n.

5. *The envelope of the family* (14). By setting the discriminant of the quadratic equation in r defined by (14) equal to zero, the following result is obtained.

Theorem 1. *Consider the* ∞^1 *integral curves of a system* S_k *which pass through a given lineal-element* E. *The* ∞^1 *osculating conic sections constructed to these curves at* E, *not only pass through* E, *but also touch the conic section*

$$(15)\qquad \begin{aligned}&[6ANX + \{n(2AT - 3CN) + (5AT - 6CN + 3BN)\}Y - 18N^2]^2\\ &\quad - 4(3EN - 4A^2)[\{3NX - (n+1)TY\}^2\\ &\quad + (n-2)\{(n+1)T^2 - 3N^2\}Y^2] = 0.\end{aligned}$$

The conic (15) is degenerate if and only if $3EN - 4A^2 = 0$, or $n = 2$. The case $n = 2$ defines the system S_0 of dynamical trajectories.

Theorem 2. *The* ∞^1 *osculating conic sections constructed at a lineal-element* E *to the* ∞^1 *integral curves of the system* S_0 *of dynamical trajectories which pass through* E, *not only pass through* E, *but also touch the two straight lines*

$$(16)\qquad \begin{aligned}&2ANX + (3AT - 4CN + BN)Y - 6N^2\\ &\quad = \pm 2(3EN - 4A^2)^{1/2}(NX - TY).\end{aligned}$$

These two straight lines intersect in the point

$$(17)\qquad Z = X + iY = \frac{6N(T + iN)}{5AT - 4CN + BN},$$

which is on the line of force.

The two straight lines (16) may be real and distinct, or real and coincident, or conjugate-imaginary.

In the Newtonian field of force where the force vector at any point is directed towards a fixed point P and the magnitude is inversely proportional to the square of the distance of the point from P, the trajectories are conic sections with one focus at P. The point (17) reduces to the point P and the straight lines (16) become the minimal straight lines tangent to the conical trajectory.

The conic sections (15) are of the form

$$(18)\qquad \alpha X^2 + 2(\beta_0 n + \beta_1)XY + (\gamma_0 n^2 + \gamma_1 n + \gamma_2)Y^2$$

$$+ 2\delta X + 2(\epsilon_0 n + \epsilon_1)Y + \eta = 0,$$

where $(\alpha, \beta, \gamma, \delta, \epsilon, \eta)$ are independent of n.

Theorem 3. *By varying k, the conic sections* (15) *form a quadratic family where n is the parameter. These conics pass through two fixed points on the line of the lineal-element E, touch a conic section, and have their centers on another conic section.*

In the first place, the points in which the conic section (18) intersect the line of E, satisfy the equations $Y = 0$, $\alpha X^2 + 2\delta X + \eta = 0$. Since these equations are independent of n, it is seen that all the conic sections of (18) pass through these two points.

The conics (18) touch the conic section

$$(19)\qquad (2\beta_0 X + \gamma_1 Y + 2\epsilon_0)^2 - 4\gamma_0(\alpha X^2 + 2\beta_1 XY + \gamma_2 Y^2 + 2\delta X + 2\epsilon_1 Y + \eta) = 0.$$

Finally the centers of the conic sections (18) are given by

$$(20)\qquad \alpha X + (\beta_0 n + \beta_1)Y + \delta = 0,$$

$$(\beta_0 n + \beta_1)X + (\gamma_0 n^2 + \gamma_1 n + \gamma_2)Y + \epsilon_0 n + \epsilon_1 = 0.$$

Thus the centers describe the conic section

$$(21)\qquad \gamma_0(\alpha X + \beta_1 Y + \delta)^2 - \beta_0(\epsilon_0 + \gamma_1 Y)(\alpha X + \beta_1 Y + \delta) - \beta_0^2(\alpha X^2 - \gamma_2 Y^2 + \delta X - \epsilon_1 Y) = 0.$$

6. *The locus of the centers of the conic sections of the family* (14). By (7), (8) and (12), it is found that the centers of the conic sections (14) are

$$(22)\qquad [(n-2)\{(n+1)T^2 - 3N^2\} + \{n(4AT - 3CN) + (7AT - 6CN + 3BN)\}r + (3EN - 5A^2)r^2]Z = 3Nr[(n+1)T - Ar + 3iN].$$

Theorem 4. *The centers of the ∞^1 osculating conic sections constructed at a lineal-element E to the ∞^1 integral curves of the system S_k, which pass through E, describe the conic section*

$$(23)\qquad (3EN - 5A^2)[3NX - (n+1)TY]^2 - A[n(4AT - 3CN) + (7AT - 6CN + 3BN)][3NX - (n+1)TY]Y + (n-2)A^2[(n+1)T^2 - 3N^2]Y^2 + 9N^2[3NX - (n+1)TY] = 0,$$

which passes through the point of E in the direction whose slope with respect to E is equal to the slope of the force vector with respect

to E multiplied by the quantity $3/(n+1)$.

The preceding result follows from equations (22) and (23)[4].

The conic section (23) is degenerate only in the cases where $A = 0$, or $n = 2$.

Theorem 5. *The centers of the* ∞^1 *osculating conic sections constructed at a lineal-element E to the* ∞^1 *integral curves of the system* S_0 *of dynamical trajectories which pass through E, describe the straight line*

$$(24) \qquad (3EN - 5A^2)X - (3ET - 4AC + AB)Y + 3N = 0.$$

This result follows from (22) by placing $n = 2$.

Theorem 6. *By varying k, the conic sections (23) form a quadratic family where n is the parameter. These conics pass through the point of E and another fixed point on the line of E, touch a conic section, and have their centers on another conic section.*

The family (23) is of the form (18) where $\eta = 0$. The remainder of the theorem follows from the proof of Theorem 3.

7. *The locus of the foci of the conic sections of the family* (14). By (7), (8) and (13), it is found that the foci of the conic sections (14) are given by the equation

$$(25) \qquad \begin{aligned}&[(n-2)\{(n+1)T^2 - 3N^2\} + \{n(4AT - 3CN)\\ &+ (7AT - 6CN + 3BN)r + (3EN - 5A^2)r^2]Z^2\\ &- 6Nr[(n+1)T - Ar + 3iN]Z - 9N^2r^2 = 0,\end{aligned}$$

together with its complex conjugate equation.

To obtain the implicit equation of the locus of the foci, we proceed in the following manner. Divide (25) by Z^2. Taking the conjugate and subtracting, the following expression

$$(26) \qquad r = -\frac{(X^2+Y^2)[3NX - (n+1)TY]}{Y[A(X^2+Y^2) - 3NX]},$$

is obtained. Substituting this value of r into (25) and simplifying, the following result may be stated.

Theorem 7. *The foci of the* ∞^1 *osculating conic sections constructed at a lineal-element E to the* ∞^1 *integral curves of the system* S_k, *which pass through E, describe the algebraic curve of sixth degree*

$$(27) \qquad \begin{aligned}&(3EN - 5A^2)(X^2+Y^2)^2[3NX - (n+1)TY]^2\\ &- [n(4AT - 3CN) + (7AT - 6CN + 3BN)]Y(X^2+Y^2)[3NX\\ &- (n+1)TY][A(X^2+Y^2) - 3NX] + (n-2)[(n+1)T^2 - 3N^2]Y^2[A(X^2+Y^2)\\ &- 3NX]^2 + 9N^2(X^2+Y^2)[3NX - (n+1)TY][2A(X^2+Y^2) - 3NX - (n+1)TY] = 0.\end{aligned}$$

This locus has double points at the circular points I and J at infinity and a singular point of fourth order at the point of E.

The case $n = 2$ is of particular interest as it gives the system S_0 of dynamical trajectories.

Theorem 8. *The foci of the* ∞^1 *osculating conic sections constructed at a lineal-element E to the* ∞^1 *dynamical trajectories, which pass through E, describe the algebraic curve of third degree*

$$(3EN - 5A^2)(X^2 + Y^2)(NX - TY) - (5AT - 6CN + BN)Y[A(X^2 + Y^2) - 3NX] + 3N^2[2A(X^2 + Y^2) - 3NX - 3TY] = 0. \tag{28}$$

This locus has two minimal asymptotes and it passes through the point of E in the direction which is the symmetrical image of that of the force vector with respect to the line of E.

The necessary and sufficient conditions that the cubic curve (28) be degenerate, are

$$6ENT - 5A^2T - 6ACN + ABN = 0, \quad T^2(3EN - 4A^2) + A^2N^2 = 0. \tag{29}$$

In this event, the cubic curve (28) consists of the straight line

$$NX + TY = 0, \tag{30}$$

and the null circle of center

$$X + iY = \frac{3NT(T + iN)}{A(T^2 + N^2)}. \tag{31}$$

The conditions (29) are realized by a Newtonian field of force. The point (31) is the center P of this particular field of force.

REFERENCES

*Presented to the American Mathematical Society, 1948.

[1] Kasner, *Differential-Geometric Aspects of Dynamics*, Amer. Math. Soc. Publications, 1913, 1934, 1948. This is referred to as the Princeton Colloquium.

[2] See the Princeton Colloquium, pp. 91-96.

[3] For the corresponding formulas relative to the osculating parabola (four-point contact), see Kasner and DeCicco, *A generalized theory of dynamical trajectories*, Trans. Amer. Math. Soc., 54, 23-38 (1943).

[4] Compare this with a corresponding result obtained by Kasner, *Systems*

of extremals in the calculus of variations, Bull. Amer. Math. Soc., 13, 289-292 (1908).

Columbia University

De Paul University

COSETS IN A SEMI-GROUP

Milo W. Weaver

INTRODUCTION. If two integers a and b have the property, $a - b$ is divisible by the integer m, a is said to be *congruent to* b *modulo* m. This is written $a \equiv b \pmod{m}$. The equality of ordinary algebra shares many of its properties with the congruence relation. In this paper, we shall suppose that the reader is familiar with this relation as discussed in the books on elementary number theory. The congruence characteristics of the integers which are prime to the modulus are well known. These integers will be called *units* in this study. One of the main objectives of this article is to present some of the properties of the integers which are not prime to the modulus.[1] Since no two of the integers $0, 1, \ldots, m - 1$ are congruent modulo m, and since every integer is congruent to one of them, they are called the *least residues* modulo m. The formula $i = km + r$, $0 \leq r < m$, gives us a method of finding the least residue of an integer i modulo m. We shall now give some examples illustrating the behavior of certain integers not prime to a composite modulus. When 30 is raised to successive powers, we obtain the set of incongruent integers modulo 360: 30, 180, 0. Similarly 12 generates the least residues: 12, 144, 288, 216, 72 modulo 360. This set contains the set: 144, 288, 216, 72, which has the properties modulo 360: each one divides the others modulo 360; $a \cdot 216 \equiv a \pmod{360}$ for each a of the subset; and 288 generates the subset. 5 generates a set of least residues modulo 360, which has the property: each divides the others modulo 360. The least residues of the units modulo 12 are: 1, 5, 7, 11. If the elements in this set are multiplied successively by 1, 2, 3, 4, 6, and 0, we get the sets of least residues modulo 12: 1, 5, 7, 11; 2, 10; 3, 9; 4, 8; 6; and 0. Furthermore these sets are disjoint and they exhaust the total set of least residues modulo 12. These examples are special cases of theorems 1, 3, and 4. Cosets in semi-groups are mentioned in theorem 1; we shall develop a theory of these cosets which is similar to the theory of cosets in groups. It is astounding that theorems 3, 4, and 6 concerning the properties of the residues modulo m are not well known; however the writer has not been able to find published proofs of these theorems. If any previous work is duplicated, then that part of the present paper can be regarded as expository.

PART 1

COSETS IN SEMI GROUPS. Let S be a set of elements denoted by small letters, $\circ$ be an operation symbol which can be placed between any two

[1]This problem as well as the problem of the generalization of the theory of cosets was suggested to the writer by H. S. Vandiver.

elements of S, and = be a relation called equal with the properties:

1. $a = a$ for each element a of S.
2. If $a = b$, then $b = a$.
3. If $a = b$ and $b = c$, then $a = c$.
4. In an operational equation $a^\circ b^\circ c^\circ \ldots {}^\circ r = t^\circ u^\circ v^\circ \ldots {}^\circ y$, any combination of elements can be replaced by its equal combination.

The symbol $^\circ$ is often omitted between two elements. If S also has the properties:

5. $ab = x$ is always solvable in S,
6. $(ab)c = a(bc)$ for each a, b, and c of S,

then S is called a *semi-group*. If furthermore:

7. $ax = b$ and $ya = b$ have solutions for each a and b of S,

then S is called a *group*.

Let S be a semi-group with a subsemi-group S'. If a_i is an element in S, then the *right spread* of a_i (written hereafter as s_i) with respect to (w.r.t.) S' is the number of elements x (which set of x's we call S'') in S' such that $a_i x = a_i$. This set is a semi-group, for $a_i x_1 x_2 = a_i x_2 = a_i$ for each two x's of S''. If S' is a finite group G, then the set S'' is a group, for it is a finite closed subset of a group. S'' can be proved to be a group even though G is infinite. S'' is said to be the *right semi-group in S w.r.t. S', belonging to a_i*. The word "left" can be substituted for "right" throughout the discussion. If a commutes with all the elements of S', we omit the word "right". These two comments apply to all future discussions of this nature. If S' is a group, it can be divided into cosets w.r.t. S''. If S is a finite semi-group containing a subsemi-group S', and c_i is an element in S, by the *left coset* $c_i S'$, we mean the set of elements obtained by multiplying c_i on the right by each element of S'. If S' is a finite group G of order n and s_i is the right spread of c_i in S w.r.t. G, let $n = s_i t_i$. Then t_i is an integer, for it is merely the index in G of the right subsemi-group in S w.r.t. G, belonging to c_i. If S is divisible uniquely into mutually exclusive left cosets w.r.t. a subsemi-group (group) S', then S' is called a *left divisor semi-group* (group) of S. The last example of the introduction illustrates this definition. We emphasize the uniqueness of the cosets, regardless of which elements we choose as coset multipliers. It follows from the definition that each element of S is divisible on the right by some element of S'.

A. A. Albert[2] and R. H. Bruck[3] discussed disjoint sets called cosets in a system called a *quasi-group*. However a quasi-group is non-associa-

[2] *Quasi-groups I*, Trans. Am. Math. Soc., Vol. 54 (1943), pp. 507-519.

[3] *Contributions to the theory of loops*, Trans. Amer. Math. Soc., Vol. 60 (1946), pp. 245-354.

tive and lacks only this property to make it a group.

A. R. Richardson[4] discussed cosets in a groupoid G. A *groupoid* is a system which is closed under a single valued binary operation. Associativity is not postulated. Richardson's definition of cosets of G w.r.t. a subset is like ours except that he permitted the presence in G of elements not divisible by any elements of G. If G is divided into cosets w.r.t. a subgroupoid B and the product of two right cosets is a right coset, then the *quotient groupoid* G/B is said to exist. Let U be the set of elements which observe the associative law, from any position, w.r.t. all the elements of G and let C be those elements of the center that are in U. Richardson proved that the quotient groupoid G/C exists.

F. W. Levi[5] discussed cosets in what he called R-semi-groups, but his definition of cosets is different from ours.

R. R. Stoll[6] proved a theorem about the division of a *finite simple semi-group* into cosets w.r.t. a subright-group. He used *coset* in the same sense that we do, but his *finite simple semigroup* cannot contain a zero element, while this is permissible in our theorem 1 which resembles Stoll's theorem.

THEOREM 1. *A finite semi-group S with a subgroup G has G as a left divisor group if and only if the right semi-group in S belonging to c with respect to G is nonvacuous for each element c in S. Furthermore if S is of order g and $c_1, c_2, \ldots, c_h$ is a complete set of coset multipliers of the left cosets of S with respect to G of order n, then $\sum_{i=1}^{h} n/s_i = g$, where s_i is the right spread of c_i*[7]. The "if" part of this theorem is a generalization of one due to Dr. H. S. Vandiver[8]. The hypothesis of his theorem is, *If the identity element of G is a right identity of S.*

To prove theorem 1, first we suppose that S is a finite semi-group with a subgroup G, and furthermore, if c is any element whatsoever in S, then the right semi-group in S belonging to c w.r.t. G is nonvacuous. We denote the identity element of G by c_1. If the coset c_1G does not exhaust S, we let c_2 be an element not in c_1G. Then c_2 is in c_2G since the right semi-group of c_2 in S w.r.t. G is nonvacuous.

[4] *Groupoids and their automorphisms*, Proc. London Math. Soc., (2) 48 (1943/45) pp. 83-111.

[5] *On semi-groups*, Bull. Calcutta Math. Soc., Vol. 36 (1944), pp. 144-146.

[6] *Representations of finite simple semi-groups*, Duke Math. Jour., Vol 11, (1944), pp. 251-265.

[7] J. L. Dorroh first proved the "if" part of this theorem, but did not publish it. The writer is not familiar with the nature of his proof.

[8] *The elements of a theory of abstract discrete semi-groups*, Vierteljahrsschrift der Naturforschenden Gesellschaft in Zurich, vol. 85 (1940), pp. 71-86.

It is clear that we can exhaust S by continuing this process. Now

$$c_i g_k = c_j g_m$$

if $k \neq m$, for if the equality held, c_j would be in $c_i G$ and c_i, in $c_j G$. The cosets are therefore mutually exclusive.

$$c_i g_k G = c_i G;$$

Hence the division of S into left cosets w.r.t. G is unique. Let c_i be any one of the c's. We denote by B_i a complete set of multipliers of the right cosets of G w.r.t. G_i', where G_i' is the right semi-group in S belonging to c_i w.r.t. G. Apparently each element in $c_i B_i$ is in $c_i G$. The converse is true also, for

$$c_i g_j = c_i g' b_k = c_i b_k,$$

where g' is in G_i' and the b's are in B. The order of $c_i B_i$ is the index of G_i' in G. Hence if g is the order of S, n is the order of G, s_i is the right spread of c_i in G, and h is the number of distinct left cosets S is divided into w.r.t. G,

$$\sum_{i=1}^{h} t_i = \sum_{i=1}^{h} n/s_i = g.$$

Next we suppose that the semi-group S has a subgroup G as a left divisor group. Let the multipliers be $c_1, c_2, \ldots,$ where c_1 is the identity of G. Then c_1 is in $c_1 G$. Let c_i be any one of the c's; then c_i is in some coset. If

$$c_i = c_k g_m,$$

where g_m is in G, then

$$c_k = c_i g_m^{-1},$$

and

$$c_i = c_i g_m^{-1} g_m = c_i c_1.$$

We conclude that c_i must be in $c_i G$ for each c_i, and furthermore c_1 is a right identity for each one of the c's and therefore for S. We have used no assumption of finiteness in this part of the proof.

If S is a semi-group such that the equation $ax = b$ is solvable for each a and b in S, S is said to be a *right-group*. This terminology was first used by A. Suschewitsch[9]. It follows, that if a semi-group

[9] *Ueber die endlichen gruppen ohne das gesetz der eindeutigen umkehrbarkeit*, Mathematische Annalen, Vol. 99 (1928), pp. 30-50.

is both a right-group and a left-group, it is a group.

THEOREM 2. *If S is a semi-group which contains a finite left divisor semi-group S' such that for some s of S, $sS' = S'$, then S' is a right-group.* Since $sS' = S'$, S' is itself a left coset; hence by the uniqueness part of the definition of left coset, $saS' = aS' = S'$ for each a of the finite semi-group S'. Therefore the equation $ax = b$ is solvable for each a and b of S' and S' is a right-group. The dual of this theorem is proved similarly. From the truth of the theorem and its dual and the fact that a semi-group which is both a right-group and a left-group is a group follows the

COROLLARY. *If S is a semi-group with a finite left divisor semi-group S' which is also a right divisor semi-group and there exist elements r and l of S such that $lS' = S'r = S'$, then S' is a group.*

PART 2

THE CONNECTION OF COSETS WITH RESIDUE CLASSES. A set of all integers which are mutually congruent modulo m is called a *residue class* modulo m. Obviously there are exactly m distinct classes modulo m., and each class contains exactly one integer of the set of least residues. These classes form an additive group and a multiplicative semi-group modulo m. Since the substitution law and the right and left distributive laws are also valid, the residue classes modulo m form a *ring*. Throughout the following discussion, we assume $m > 1$. If C is a least residue such that each prime divisor of m divides C, we call the residue class which is congruent to C one of the *first type*. If D is a least residue such that some prime divisors of m, but not all of them divide D, we call the residue class which is congruent to D one of the *second type*. These two types, together with the units, exhaust the residue classes modulo m. A residue class which is made up of units is called a *unit* also.

Let $m = p_1^{i_1} p_2^{i_2} \dots p_j^{i_j}$, where the p's are distinct primes and let r be a residue class of the first type whose least residue is A. Write A in the form, $p_1^{k_1} p_2^{k_2} \dots p_j^{k_j} \cdot D$, where $(D, m) = 1$, $k_v > 0$. If b is the smallest integer such that $bk_t \geq i_t$ for each t in the range $1 \geq t \geq j$, then b is the smallest value of x such that $r^x \equiv 0 \pmod m$ and is called the *nullifying exponent* of r modulo m. Now suppose

$$r^y \equiv r^z \pmod m,$$

where $b > y > z > 0$. Then

$$r^z(r^{y-z} - 1) \equiv 0 \pmod m.$$

Therefore

$$r^{y-z} \equiv 1 \pmod{m_1},$$

where $m_1 = m/(A^z, m)$; then either $m_1 = 1$ or m_1 is the product of certain

prime divisors of m. But $m_1 \neq 1$, for this would imply $r^z \equiv 0 \pmod{m}$, which would make $z \geq b$. It is known that if $(r,m) \neq 1$, $r^x \not\equiv 1 \pmod{m}$ for any $x > 0$; hence m_1 is not a multiple of prime divisors of m and $r^y \not\equiv r^z \pmod{m}$. We have proved

THEOREM 3. *If b is defined as above, then b is the nullifying exponent of a residue class r of the first type; furthermore r generates a semi-group of order b, and the zero residue class is the first repeated one.*

Let C be the least residue of a residue class r modulo m of the second type such that $C = p_1^{h_1} p_2^{h_2} \ldots p_c^{h_c} \cdot D$, where $h_v > 0$, $(D, m) = 1$. Let $A = p_1^{i_1} p_2^{i_2} \ldots p_c^{i_c}$; $m = AB$, r belong to n modulo m, and finally let b be the nullifying exponent of r modulo A. Hence $(B, p_1 p_2 \ldots p_c) = 1$. Also $r^n \equiv 1 \pmod{B}$, and $r^b \equiv 0 \pmod{A}$. Therefore

$$r^{b+n} \equiv r^b \pmod{AB}.$$

If $r^p \equiv r^q \pmod{m}$ for $b + n \geq p > q \geq 0$, then $r^p - r^q \equiv 0 \pmod{m}$, and

$$r^q(r^{p-q} - 1) \equiv 0 \pmod{B}.$$

Hence $r^{p-q} \equiv 1 \pmod{B}$, and $p - q$ is a multiple of n.

Let

$$p - q = k_1 n, \quad k_1 \geq 1.$$

Then

$$r^q(r^{k_1 n} - 1) \equiv 0 \pmod{p_1^{i_1} p_2^{i_2} \ldots p_c^{i_c}}.$$

But $C^{k_1 n} \not\equiv 1 \pmod{p_t}$, $1 \leq t \leq c$, by the known theorem mentioned in the last part of the proof of theorem 3. Therefore

$$r^q \equiv 0 \pmod{A}.$$

and

$$q = k_2 + b, \quad k_2 \geq 0.$$

We can therefore write

$$p = q + k_1 n = k_2 + b + k_1 n;$$

but $p \leq b + n$. Hence $k_2 = 0$ and $k_1 = 1$, and the distinct elements of the semi-group generated by r are: $r, r^2, \ldots, r^{b+n-1}$. The residue classes $r^b, r^{b+1}, \ldots, r^{b+n-1}$ form a cyclic group under multiplication[10].

[10] F. C. Biesele, *An introduction to the theory of semi-groups*, Master's Thesis, Univ. of Texas, (1933), p. 9.

THEOREM 4. *If r is a residue class of the second type, it generates a semi-group modulo m of order $b+n-1$, which contains as a subgroup the residue classes:*

$$r^b,\ r^{b+1},\ \ldots,\ r^{b+n-1},$$

where b and n are defined in the proof above.

Semi-groups of the type mentioned in theorems 3 and 4 are called *cyclic*. Their importance in general systems has been discussed[11].

Type two contains as an interesting sub-type $2a$, the set of residue classes whose least positive residues contain as divisors the highest powers of certain prime divisors of m, but are prime to other prime divisors of m. Concerning this type, we have as a corollary to theorem 4, the

COROLLARY. *If r is a residue class of type $2a$, it generates a cyclic group of order n, where n is defined in the proof of theorem 2.* This follows from the proof of the theorem, where $b = 1$.

If we denote the group of units modulo m by G, then by theorem 1 and its dual, G is both a left and right divisor group of the semi-group S of all the residue classes modulo m. Furthermore, $\sum_{i=1}^{h} t_i = m$. We shall obtain two methods for getting the t's and the s's connected with the coset multipliers. Let a_i be any element of S and x be any element of G such that

$$a_i x \equiv a_i (\text{mod } m).$$

then

$$x \equiv 1 \ (\text{mod } m/(a_i, m)).$$

This gives us a way of finding s_i and therefore t_i, since $t_i = n/s_i$. On the other hand, if g_1 and g_2 are two elements in G such that

$$a_i g_1 \equiv a_i g_2 \ (\text{mod } m),$$

then

$$g_1 \equiv g_2 \ (\text{mod } m/(a_i, m)),$$

which gives us a way of finding the distinct elements of the coset $a_i G$, and therefore t_i and s_i.

Let the semi-group S of residue classes modulo m contain S' as a left divisor semi-group. Then since S has an identity e such that $eS' = S'$, and since S' is also a right divisor semi-group, S' is a group by the corollary to theorem 2. But by what we proved in the second part

[11] H. S. Vandiver, Bull. Am. Math. Soc., Vol. 40 (1934), pp. 914-920; Am. Math. Monthly, Vol. 46, (1939), pp. 22-26.

of the proof of theorem 1, the identity of S' is an identity for each of the elements of S. Since $a^r \equiv 1 \pmod{m}$ implies $(a, m) = 1$, the group of units modulo m contains each subgroup of S which has 1 as its identity. This proves

THEOREM 5. *A semi-group S of residue classes modulo m contains no left divisor semi-group not contained in the subgroup of units.*

PART 3

FACTORIZATION IN THE MULTIPLICATIVE SEMI-GROUP OF RESIDUE CLASSES MODULO m. The elements of the ring modulo m which are not units are called *non-units*. The units are denoted by g's. Their least residues are denoted by u's, v's, and w's. If G is the set of units and a is a non-unit, then the elements of the coset aG are called *associates* of one another. A *Prime* is a non-unit residue class which contains no divisors other than itself and its associates. Capital letters are used to distinguish Primes from the primes of arithmetic. Evidently the product of units is a unit and if any factor of a product is a non-unit, the product is also. A set of elements are called *Relatively Prime* if no two of them have a common non-unit divisor. H. S. Vandiver[12] proved that the distinct Primes modulo m are the elements whose least residues are prime divisors of m, together with their associates. The Primes modulo m are denoted by capital letters, P, Q, R, and S, while the arithmetic primes are denoted by the small letters, p, q, r, and s.

Since the least residues modulo m represent a complete set of residue classes, each element of the ring of residue classes modulo m can be factored into Primes and units. Suppose two factorizations of the non-unit n_k into Primes and units are $P_1P_2 \ldots P_c \cdot g_1$ and $Q_1Q_2 \ldots Q_d \cdot g_2$. By commuting the factors if necessary and using the definition of associates, we can write these in the forms

$$(1) \qquad R_1^{a_1} R_2^{a_2} \ldots R_x^{a_x} \cdot g_4 \quad \text{and} \quad S_1^{b_1} S_2^{b_2} \ldots S_y^{b_y} \cdot g_5$$

where the R's are Relatively Prime and likewise the S's. If each of the R^a's is an associate of an S^b and conversely, then $x = y$. And n_k is said to have a *unique factorization* into Primes and units.

LEMMA. *If p^b, for $b > 0$, is the largest power of the prime p which divides m and if p is the least residue of P, then no two distinct elements of the set $P, P^2, \ldots, P^b$ are associates. If there are other elements in the semi-group generated by P, they are associates of P^b.*

If P^j and P^i are associates, where $j \leq i \leq b$, then

$$P^i \equiv gP^j \pmod{m},$$

and

[12] *Theory of finite algebras*, Transactions of the American Mathematical Society, Vol. 13, (1912) pp. 293-304.

$$P^j(P^{i-j} - g) \equiv 0 \pmod{m}.$$

Hence

$$g^{-1}P^{i-j} \equiv 1 \pmod{m/(p^j, m)},$$

and P^{i-j} is a unit modulo $m/(p^j, m)$. But since $j \leq i \leq b$ it follows that $i = j$, and no two distinct elements of the set P, P^2, ..., P^b are associates. To prove the second part of the theorem, we note first that if $m = p^b$, the set mentioned above exhausts the semi-group generated by P. If $m = p^b m_1$, $m_1 \neq 1$, let $u_1 = m_1 + p$ and p belong to n modulo m_1. Then $(u_1, m_1) = (u_1, p) = 1$, and u_1 is the least residue of a unit modulo m. Therefore

$$p^{b+1} \equiv u_1 p^b \pmod{m},$$

and

$$p^{b+i} \equiv (u_1)^i p^b \pmod{m}$$

for $i > 1$. Hence the elements of the set P^z, $b \leq z \leq b + n - 1$ are all associates. We note that n divides the order of the group of units modulo m/p^b.

THEOREM 6. *Each of the non-unit elements of the ring of residue classes modulo m has a unique factorization into Primes and units.* Any two factorizations[13] of a non-unit n_k into Primes and units can be written in the forms (1). Since each of the R's and S's is congruent to an element of the type pv, where p is a prime divisor of m and v is the least residue of a unit, we can write (1) as

$$(2) \qquad (r_1 v_1)^{a_1}(r_2 v_2)^{a_2}\ldots(r_x v_x)^{a_x}\cdot u_4 \text{ and}$$
$$(s_1 w_1)^{b_1}(s_2 w_2)^{b_2}\ldots(s_y w_y)^{b_y}\cdot u_5,$$

where the r's and s's are prime divisors of m and the v's and w's are prime to m. Since the members of (2) are factorizations of n_k, they are congruent; and each r is some s and conversely. Hence $x = y$ and (2) may be written as

$$(3) \qquad (r_1 v_1)^{a_1}(r_2 v_2)^{a_2}\ldots(r_x v_x)^{a_x}\cdot u_4 \text{ and}$$
$$(r_1 w_1)^{b_1}(r_2 w_2)^{b_2}\ldots(r_x w_x)^{b_x}\cdot u_5.$$

[13]The number of distinct solutions of the congruence $a \equiv x_1 x_2 \cdots x_{r+1}$ (mod m) was discussed by B. Gyires, *Uber die faktorisation im restklassenring mod m*, Publicationes Mathematicae, Vol. 1 (1949), pp. 51-55. But Gyires did not discuss unique factorization.

It follows from the lemma that if $a_i \geq b_i \geq b$ where b is the largest power of r_i dividing m, then $(r_i v_i)^{a_i}$ is an associate of $(r_i w_i)^{b_i}$. On the other hand if a_j and b_j are exponents such that $a_j \leq b_j \leq b$, where r_j^b is the largest power of r_j which divides m, we can write a congruence from the two members of (3):

$$r_j^{a_j} E \equiv r_j^{b_j} F \pmod{m},$$

where $(r_j, E) = (r_j, F) = 1$. Hence

$$E \equiv r_j^{b_j - a_j} F \pmod{m/(r^{a_j}, m)},$$

and $b_j = a_j$ since r_j divides $m/(r^{a_j}, m)$ for $a_j < b$. Hence n_k has a unique factorization into Primes and units.

PART 4

SOME GENERALIZATIONS OF THEOREMS CONCERNING COSETS IN GROUPS. A subgroup G' of a group G is said to be *invariant* in G if $a^{-1}G'a = G'$ for each a in G. Obviously G' is invariant in G if and only if $aG' = G'a$ for each a in G. A subsemi-group S' of a semi-group S is said to be *invariant* in S if $aS' = S'a$ for each a in S. If S_1 and S_2 are subsets of S, by S_1S_2 we mean the set of elements $s_i s_j$ where s_i and s_j range independently over S_1 and S_2, respectively.

THEOREM 7. *If a semi-group is divided into cosets with respect to a left divisor group G, a necessary and sufficient condition that two left cosets aG and bG be identical is that $a = bg$, where g is in G.* Suppose that $a = bg$, where g is in G. Then a is in bG. But a is in aG, since the identity of G is a right identity of the given semi-group. Hence aG is bG, since the cosets are mutually exclusive. On the other hand if the cosets aG and bG are identical, then since the identity of G is a right identity for a and b also, a and b are both in bG. Then $a = bg$, where g is in G.

The cosets of an invariant subgroup of a group form a group of cosets. Similarly it is obvious that if a semi-group S contains an invariant subsemi-group S' such that $S'S' = S'$, then all the cosets of S' in S form a semi-group under coset multiplication.

If a left divisor subsemi-group of a semi-group S is invariant in S, we call it a *divisor semi-group*.

THEOREM 8. *Let S be a semi-group which contains a divisor group G. Then if the coset semi-group S/G is a group, S is itself a group.* Let $c_1, c_2, \ldots, c_m$ be a complete set of coset multipliers. G is the identity coset of S/G; and since S/G is a group, each c_iG has a unique inverse c_jG such that

$$c_iG \cdot c_jG = c_jG \cdot c_iG = c_ic_jG = c_jc_iG = G.$$

Hence $c_ic_j = g_1$ and $c_jc_i = g_2$ where g_1 and g_2 are both in G, and we may write $c_i(c_jg^{-1}) = e$ and $(g^{-1}c_j)c_i = e$ where e is the identity of G and therefore of S. It is known that if a semi-group has an identity and every element has an inverse, then the semi-group is a group. Hence since each element of S can be used as a coset multiplier, S is a group.

COROLLARY. *Let S be a semi-group with a divisor group G such that the coset semi-group S/G is a group. Then for each coset multiplier c_i of a complete set of coset multipliers $c_1, c_2, \ldots, c_m$, there is a unique c_j and a unique c_k of the c's such that c_ic_j and c_kc_i are elements of G. In fact c_j is c_k.* We showed the existence of a c_j satisfying both the condition on c_j and c_k in the proof of the last theorem. Suppose there is a c_j' such that

$$c_ic_j' = g_3,$$

where g_3 is in G. Then, using the notation in the proof of theorem 8, and the fact that S is a group

$$c_j = c_j'g_4,$$

where g_4 is in G. By theorem 7, c_j' is c_j. Similarly c_i has only one left multiplier among the c's such that the product is in G.

An element is said to be *homogeneous*[14] in a semi-group S if it is the product of two elements of S. If every element of S is homogeneous in S, S is said to be homogeneous. It is clear that an element of a subsemi-group S' of S which is homogeneous in S' is also homogeneous in S, but the converse is not necessarily true. Obviously, if S contains a left divisor semi-group, S is homogeneous and furthermore $SS = S$. If I is a subset of a semi-group S such that SI is contained in I, then I is called a *left ideal* of S. If I is both a left and right ideal of S, it follows from the definition that I is a semi-group.

Suppose a semi-group S contains a finite homogeneous subsemi-group S' as a left divisor semi-group. Choose any element c_j of S' as a coset multiplier. Then

$$c_jS' = c_js'S'$$

for each c_js' of c_jS'. Hence c_jS' is a right ideal of S'. Since S' is a homogeneous left divisor semi-group of S, every element of S' is in a left coset with an element of S' as a multiplier, and we can exhaust S' by such cosets in such a way that they are disjoint. Furthermore, since S' is a semi-group, none of these cosets contain any elements outside of S'. We state our results as

THEOREM 9. *Let a semi-group contain a finite homogeneous subsemi-*

[14] A. R. Richardson, *Groupoids and their automorphisms*, Proceedings of the London Mathematical Society, Vol. 48 (1943/45), p. 96.

group S' as a left divisor semi-group. Then S' consists of a set of disjoint right ideals.

Since S' is homogeneous, $S'S' = S'$. If S' is also invariant in the given semi-group it follows from the remark before theorem 8 that the cosets of S w.r.t. S' form a semi-group. Therefore

$$I_m I_n = I_m = I_n$$

for each pair of the coset ideals. Therefore, there is only one coset of S' and that is S' itself. Consequently

$$s'S' = S' = S's'$$

for each s' of S' and S' is a group. This proves the

COROLLARY. *If a semi-group contains S' as a finite homogeneous divisor semi-group, then S' is a group.*

It is known that if a group G contains a subgroup G' such that G/G' is a group, then G' is invariant in G. An obvious extension to semi-groups is: if the semi-group S contains a left and right divisor group G such that the product of two left cosets is a left coset and the product of two right cosets is a right coset, then G is invariant in S.

University of Texas

POWERS OF SUMS AND SUMS OF POWERS

Pedro A. Piza

Since ancient times the sums of the powers of positive integers and the study of the properties of power sums has attracted the attention of arithmeticians. Chapter XIV of Edouard Lucas' *Théorie des Nombres*, published in Paris in 1891, contains an excellent survey of this general theory and its history.

Many notable relations concerning the sums of powers are expounded in Lucas' book, due to a galaxy of mathematicians of all the ages including Fibonacci, Fermat, Pascal, Newton, Bachet, Fuler, Roberval, Abel, Jacobi, Radicke, Adams, Staudt, Clausen, Stern, Genocchi, Lacroix, Cesaro, Stirling, Boole, and Lucas himself.

Particularly celebrated are Jacques Bernoulli's classical formulas, published in 1712 in *Ars Conjectandi*, wherein the values of $\sum_{a=1}^{x} a^n$ are given as power functions of x, such as

$$\sum_{a=1}^{x} a^6 = (1/7)x^7 + (1/2)x^6 + (1/2)x^5 - (1/6)x^3 + (1/42)x.$$

$$\sum_{a=1}^{x} a^7 = (1/8)x^8 + (1/2)x^7 + (7/12)x^6 - (7/24)x^4 + (1/12)x^2.$$

In this paper we propose to present and to prove some fundamental theorems on power sums, which this author considers to be new, by which Bernoulli's formulas and other known attributes of the sums of powers are easily derived, and new properties developed.

We begin by stating and demonstrating

Theorem A

The n-th power of the sum of the first x integers, is equal to the arithmetical mean of the 2^{n-1} sums of powers contained in the summation

$$\sum_{b=1}^{\infty} \binom{n}{2b-1} \sum_{a=1}^{x} a^{2n+1-2b}.$$

This theorem is formulated as follows:

$$(1).. \qquad \left(\sum_{a=1}^{x} a\right)^n = \frac{\sum_{b=1}^{\infty} \binom{n}{2b-1} \sum_{a=1}^{x} a^{2n+1-2b}}{2^{n-1}},$$

which formula is to be proved to constitute an identity for arbitrary values of the integers n and x.

It is known that $\sum_{a=1}^{x} a = x(x+1)/2 = (x^2+x)/2$.

Hence (1) may be written in the form

$$(2).. \qquad (x^2 + x)^n = 2\sum_{b=1}^{\infty}\binom{n}{2b-1}\sum_{a=1}^{x} a^{2n+1-2b}.$$

Let n be any positive integer. When $x = 1$,

$$\sum_{a=1}^{x} a^{2n+1-2b} = 1$$

for all values of n and b. Therefore we are to prove that

$$(3).. \qquad 2^n = 2\sum_{b=1}^{\infty}\binom{n}{2b-1}.$$

We have $$(1+1)^n = \sum_{b=1}^{\infty}\binom{n}{2b-2} + \sum_{b=1}^{\infty}\binom{n}{2b-1} = 2^n$$

$$(1-1)^n = \sum_{b=1}^{\infty}\binom{n}{2b-2} - \sum_{b=1}^{\infty}\binom{n}{2b-1} = 0$$

Subtracting we get (3).

Suppose that (2) is true when $x = z - 1$, $z > 2$, so that

$$(z^2 - z)^n = 2\sum_{b=1}^{\infty}\binom{n}{2b-1}\sum_{a=1}^{z-1} a^{2n+1-2b}.$$

Add to each member of this equation the function

$$2\sum_{b=1}^{\infty}\binom{n}{2b-1} z^{2n+1-2b}$$

and we obtain immediately

$$(z^2 + z)^n = 2\sum_{b=1}^{\infty}\binom{n}{2b-1}\sum_{a=1}^{z} a^{2n+1-2b},$$

which proves that (2) is also true when $x = z$. This completes the demonstration of Theorem A by mathematical induction.

If for the sake of brevity the Lucas notation for arbitrary x

$$\sum_{a=1}^{x} a^n = S_n$$

is used, we express formula (2) in the form of identities for the first

few values of n, as follows:

$$
\begin{aligned}
(4)..\quad (x^2+x)/2 &= S_1\\
(x^2+x)^2/2 &= 2S_3\\
(x^2+x)^3/2 &= 3S_5 + S_3\\
(x^2+x)^4/2 &= 4S_7 + 4S_5\\
(x^2+x)^5/2 &= 5S_9 + 10S_7 + S_5\\
(x^2+x)^6/2 &= 6S_{11} + 20S_9 + 6S_7\\
(x^2+x)^7/2 &= 7S_{13} + 35S_{11} + 21S_9 + S_7\\
(x^2+x)^8/2 &= 8S_{15} + 56S_{13} + 56S_{11} + 8S_9\\
(x^2+x)^9/2 &= 9S_{17} + 84S_{15} + 126S_{13} + 36S_{11} + S_9\\
(x^2+x)^{10}/2 &= 10S_{19} + 120S_{17} + 252S_{15} + 120S_{13} + 10S_{11}
\end{aligned}
$$

and so forth.

Since $S_3 = (S_1)^2$, we are able to state the following corollary

Theorem B

The n-th power of the sum of the first x cubes, is equal to the arithmetical mean of the 2^{2n-1} sums of powers contained in the summation

$$\sum_{b=1}^{\infty} \begin{bmatrix} 2n \\ 2b-1 \end{bmatrix} \sum_{a=1}^{x} a^{4n+1-2b}.$$

We formulate this theorem thus:

$$(5)..\quad \left(\sum_{a=1}^{x} a\right)^{2n} = \left(\sum_{a=1}^{x} a^3\right)^n = \frac{\sum_{b=1}^{\infty} \begin{bmatrix} 2n \\ 2b-1 \end{bmatrix} \sum_{a=1}^{x} a^{4n+1-2b}}{2^{2n-1}}.$$

There is an infinitude of such relations corollary to Theorem A. For instance

$$\left(\sum_{a=1}^{x} a\right)^{3n} = \left[\frac{\sum_{a=1}^{x} (a^3 + 3a^5)}{4}\right]^n = \frac{\sum_{b=1}^{\infty} \begin{bmatrix} 3n \\ 2b-1 \end{bmatrix} \sum_{a=1}^{x} a^{6n+1-2b}}{2^{3n-1}}.$$

Theorems A and B concern sums of odd powers. We have also found similar relations concerning the sums of even powers, which may be stated as

Theorem C

$(2x + 1)$ times the n-th power of the sum of the first x integers, is equal to the arithmetical mean of the $3\cdot 2^{n-1}$ sums of powers contained in the summations

$$\sum_{b=1}^{\infty}\left[\binom{n}{2b-3}+\binom{n+1}{2b-2}\right]\sum_{a=1}^{x} a^{n-2+2b}, \text{ when } n \text{ is even, or}$$

$$\sum_{b=1}^{\infty}\left[\binom{n}{2b-2}+\binom{n+1}{2b-1}\right]\sum_{a=1}^{x} a^{n-1+2b}, \text{ when } n \text{ is odd.}$$

For $n = 2r$ and $n = 2r + 1$, Theorem C is formulated respectively as follows:

(6).. $$(2x+1)(x^2+x)^{2r} = 2\sum_{b=1}^{\infty}\left[2\binom{2r}{2b-3}+\binom{2r}{2b-2}\right]\sum_{a=1}^{x} a^{2r-2+2b}.$$

(7).. $$(2x+1)(x^2+x)^{2r+1} = 2\sum_{b=1}^{\infty}\left[2\binom{2r+1}{2b-2}+\binom{2r+1}{2b-1}\right]\sum_{a=1}^{x} a^{2r+2b}.$$

We shall prove (6) by induction with respect to x, and it will be obvious that (7) can be proved in a similar manner. Let r be any integer. When $x = 1$, (6) becomes

$$3\cdot 2^{2r} = \sum_{b=1}^{\infty}\left[4\binom{2r}{2b-3}+2\binom{2r}{2b-2}\right]$$

$$= 4\cdot 2^{2r-1} + 2\cdot 2^{2r-1} = 2^{2r+1} + 2^{2r} = 2^{2r}(2+1).$$

Suppose that (6) is true when $x = z - 1$, $z > 2$, so that

$$(2z-1)(z^2-z)^{2r} = 2z(z-z^2)^{2r} - (z-z^2)^{2r}$$

$$=2\binom{2r}{0}z^{2r+1} - 2\binom{2r}{1}z^{2r+2} + 2\binom{2r}{2}z^{2r+3} - 2\binom{2r}{3}z^{2r+4} + 2\binom{2r}{4}z^{2r+5} - \dots$$

$$-\binom{2r}{0}z^{2r} + \binom{2r}{1}z^{2r+1} - \binom{2r}{2}z^{2r+2} + \binom{2r}{3}z^{2r+3} - \binom{2r}{4}z^{2r+4} + \dots$$

$$= \sum_{b=1}^{\infty}\left[4\binom{2r}{2b-3}+2\binom{2r}{2b-2}\right]\sum_{a=1}^{z-1} a^{2r-2+2b}.$$

Add to each member the function

$$\sum_{b=1}^{\infty}\left[4\binom{2r}{2b-3}+2\binom{2r}{2b-2}\right]z^{2r-2+2b},$$

and right away we get

$$2z(z+z^2)^{2r}+(z+z^2)^{2r}=(2z+1)(z^2+z)^{2r}$$

$$=2\sum_{b=1}^{\infty}\left[2\binom{2r}{2b-3}+\binom{2r}{2b-2}\right]\sum_{a=1}^{z}a^{2r-2+2b},$$

which proves that (6) is true when $x = z$, completing the demonstration.

[To prove (7) in the same way, write its first member, when $x = z - 1$, in the form $(1 - 2z)(z - z^2)^{2r+1}$.]

For the first few values of r, the following identities are obtained with formulas (6) and (7):

$$\begin{aligned}
(8)..\quad (2x+1)(x^2+x)/2 &= 3S_2\\
(2x+1)(x^2+x)^2/2 &= S_2+5S_4\\
(2x+1)(x^2+x)^3/2 &= 5S_4+7S_6\\
(2x+1)(x^2+x)^4/2 &= S_4+14S_6+9S_8\\
(2x+1)(x^2+x)^5/2 &= 7S_6+30S_8+11S_{10}\\
(2x+1)(x^2+x)^6/2 &= S_6+27S_8+55S_{10}+13S_{12}\\
(2x+1)(x^2+x)^7/2 &= 9S_8+77S_{10}+91S_{12}+15S_{14}
\end{aligned}$$

and so forth.

It is clear that we are able to solve recurrently for any S_n as a function of x, by considering the identities in (4) and in (8) as simultaneous equations. Better still they can be solved as functions of $x^2 + x = y$. For instance with (4) we get

$$\begin{aligned}
S_3 &= (1/4)y^2\\
3S_5 &= (1/2)y^3-(1/4)y^2\\
S_5 &= (1/6)y^3-(1/12)y^2\\
4S_7 &= (1/2)y^4-4[(1/6)y^3-(1/12)y^2]\\
S_7 &= (1/8)y^4-(1/6)y^3+(1/12)y^2.
\end{aligned}$$

and so forth.

Upon substituting $x^2 + x$ for y we now get S_7 as a function of x as follows:

$$S_7=(1/8)(x^2+x)^4-(1/6)(x^2+x)^3+(1/12)(x^2+x)^2$$

$$=(1/8)x^8+(1/2)x^7+(7/12)x^6-(7/24)x^4+(1/12)x^2,$$

which is Bernoulli's formula.

In the same manner the identities in (8) may be solved as simultaneous equations to get any S_{2r} as a function of x and of y. For instance

$$\begin{aligned} S_2 &= (2x+1)[\ (1/6)y\ ] \\ 5S_4 &= (2x+1)[\ (1/2)y^2 - (1/6)y] \\ S_4 &= (2x+1)[(1/10)y^2 - (1/30)y] \\ 7S_6 &= (2x+1)[\ (1/2)y^3 - (1/2)y^2 + (1/6)y] \\ S_6 &= (2x+1)[(1/14)y^3 - (1/14)y^2 + (1/42)y] \end{aligned}$$

and so forth.

Again substituting $x^2 + x$ for y, Bernoulli's formula is obtained:

$$S_6 = (2x+1)[(1/14)(x^2+x)^3 - (1/14)(x^2+x)^2 + (1/42)(x^2+x)]$$

$$= (1/7)x^7 + (1/2)x^6 + (1/2)x^5 - (1/6)x^3 + (1/42)x.$$

Note that by making $n = 2$ in formula (5) we have

$$2(S_3)^2 = S_5 + S_7,$$

which is a relation first found by Jacobi in 1863, of which formula (5) is a generalization. (See Lucas' *Théorie des Nombres*, p. 233).

Also by squaring the formula

$$S_2 = (2x+1)(1/6)y$$

we get

$$(S_2)^2 = (2x+1)^2(1/36)y^2 = (4y+1)(1/36)y^2 = (1/9)y^3 + (1/36)y^2.$$

Substituting now the values in (4)

$$y^2 = 4S_3, \quad y^3 = 6S_5 + 2S_3,$$

we obtain

$$(S_2)^2 = [6S_5 + 3S_3]/9.$$

Hence

$$3(S_2)^2 = 2S_5 + S_3,$$

which is again another formula given by Lucas.

By expressing $(y^4 + y^2)^2/4 = (y^4 - y^2)^2/4 + (2y^3)^2/4$ in terms of sums of powers, a family of Pythagorean triangles can be formulated thus:

$$(2S_7 + 2S_5 + S_3)^2 = (2S_7 + 2S_5 - S_3)^2 + (3S_5 + S_3)^2.$$

THE NON-EUCLIDEAN PROJECTILE

Curtis M. Fulton

The principal object of this paper is to show that the path of a projectile in the Hyperbolic Space is, in general, a parabola. For this purpose it is essential to develop expressions for the acceleration in a suitable coordinate system.

Through a fixed point O, let there be given three directed axes, numbered 1, 2, 3, mutually perpendicular. We define the direction angles α, β, γ of a radius vector OP in the usual way. Then $\sinh OP \cos \alpha$, $\sinh OP \cos \beta$, $\sinh OP \cos \gamma$ will be the Weierstrass' coordinates of P and the direction cosines satisfy the relation

$$(1) \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

[2, p. 68 and 5, p. 86]. Now we introduce limiting surface coordinates x, y, z as follows:

$$(2) \qquad e^{-x} = \cosh OP - \sinh OP \cos \alpha, \quad e^{-x}y = \sinh OP \cos \beta,$$

$$e^{-x}z = \sinh OP \cos \gamma$$

[Cf. 4, p. 97 and 8, p. 165]. Here x = const. represents a family of limiting surfaces or horospheres that intersect the l-axis perpendicularly at a point whose directed distance from O is x. The second family, y = const., consists of planes perpendicular to the 12-plane and parallel to the 31-plane. Similarly, z = const. is a family of planes perpendicular to the 31-plane and parallel to the 12-plane. We thus have a triply orthogonal system of surfaces which intersect in limiting curves and parallels to the l-axis respectively. The variable x measures distances along these parallels, while $e^{-x}y$ and $e^{-x}z$ determine directed distances on the respective limiting curves.

Let $P(x, y, z)$ and $P_1(x_1, y_1, z_1)$ be any two points and consider the directed line segment PP_1. We now imagine three axes through P, tangent to the coordinate curves and therefore mutually perpendicular. Generalizing (2) we obtain the following relations, involving the direction cosines with respect to the new system

$$(3) \qquad e^{-(x_1-x)} = \cosh PP_1 - \sinh PP_1 \cos \alpha,$$

$$(4) \qquad e^{-x_1}(y_1 - y) = \sinh PP_1 \cos \beta,$$

$$e^{-x_1}(z_1 - z) = \sinh PP_1 \cos \gamma.$$

Hence, by a simple manipulation, because of (1),

(5) $\cosh PP_1 = \cosh(x_1 - x) + 1/2\ e^{-x_1 - x}[(y_1 - y)^2 + (z_1 - z)^2]$,

(6) $$\sinh PP_1 \cos \alpha = \sinh (x_1 - x)$$
$$+ 1/2\ e^{-x_1 - x}[(y_1 - y)^2 + (z_1 - z)^2].$$

The equations $x = x(t)$, $y = y(t)$, $z = z(t)$, where t is time, determine the motion of a particle in a curve and we shall assume, that these functions are twice differentiable. If P and P_1 correspond to t and $t + \Delta t$, respectively, we divide (5) by Δt^2 and find an expression for $\lim\limits_{\Delta t \to 0} \dfrac{PP_1}{\Delta t}$ which we denote by $\dot{s}$. Thus,

(7) $$\dot{s}^2 = \dot{x}^2 + e^{-2x}(\dot{y}^2 + \dot{z}^2),$$

the dots indicating differentiation with respect to t. Clearly, $\dot{s}$ will be the magnitude of the velocity and integration will yield the arc length s. The limiting position of PP_1 is found dividing (6) and (4) by Δt and passing to the limit as $\Delta t \to 0$. By this procedure we have for the direction cosines of the tangent line, which we indicate by means of the subscript T,

(8) $$\dot{s} \cos \alpha_T = \dot{x},\quad \dot{s} \cos \beta_T = e^{-x}\dot{y},\quad \dot{s} \cos \gamma_T = e^{-x}\dot{z}.$$

In order to define acceleration, we determine the point $P_2(x_2, y_2, z_2)$ on the tangent to the curve at P so, that $PP_2 = \Delta t\dot{s}$ and the direction of PP_2 coincides with that of increasing values of t. Using (3) and (4) and taking the direction cosines from (8), we see that the coordinates of P_2 are given by

$$e^{-(x_2 - x)} = \cosh (\Delta t\dot{s}) - \frac{\dot{x}}{\dot{s}} \sinh (\Delta t\dot{s}),$$

$$y_2 - y = \frac{\dot{y} \sinh (\Delta t\dot{s})}{\dot{s} \cosh (\Delta t\dot{s}) - \dot{x} \sinh (\Delta t\dot{s})},$$

$$z_2 - z = \frac{\dot{z} \sinh (\Delta t\dot{s})}{\dot{s} \cosh (\Delta t\dot{s}) - \dot{x} \sinh (\Delta t\dot{s})}.$$

Now, making use of these equations and, where necessary, of (7), the existence of the following limits may be shown without difficulty:

$$\lim_{t \to 0} \frac{2(x_1 - x_2)}{\Delta t^2} = \ddot{x} + e^{-2x}(\dot{y}^2 + \dot{z}^2),$$

$$\lim_{\Delta t \to 0} \frac{2(y_1 - y_2)}{t^2} = \ddot{y} - 2\dot{x}\dot{y},$$

$$\lim_{\Delta t \to 0} \frac{2(z_1 - z_2)}{t^2} = \ddot{z} - 2\dot{x}\dot{z}.$$

Also, because of (5), the existence of

$$\lim_{\Delta t \to 0} \frac{2P_2P_1}{t^2} = a$$

will be secured and we define a to be the acceleration. Let us forego the explicit expression for a and write the direction cosines of the limiting position of the directed line segment P_2P_1. Using (6) and (4) we have

$$\text{(9)} \qquad \begin{aligned} a \cos\alpha_A &= \ddot{x} + e^{-2x}(\dot{y}^2 + \dot{z}^2), \\ a \cos\beta_A &= e^{-x}(\ddot{y} - \dot{x}\dot{y}), \\ a \cos\gamma_A &= e^{-x}(\ddot{z} - \dot{x}\dot{z}), \end{aligned}$$

the subscript A indicating the direction of the acceleration.

The above results may be obtained from an entirely different point of view in Differential Geometry, postulating Lagrange's equations of motion [3, p. 106 and 6, p. 101]. In [4, p. 127] no clear definition of acceleration seems to appear. If in our definition we especially set $t = s$, we are led to the curvature $\frac{1}{R}$ [1, p. 603]. Using (8) and (9) we find by familiar methods

$$\begin{aligned} a \cos\alpha_A &= \frac{s^2}{R}\cos\alpha_N + \ddot{s}\cos\alpha_T, \\ a \cos\beta_A &= \frac{s^2}{R}\cos\beta_N + \ddot{s}\cos\beta_T, \\ a \cos\gamma_A &= \frac{s^2}{R}\cos\gamma_N + \ddot{s}\cos\gamma_T, \end{aligned}$$

where the subscript N indicates the direction of curvature, which is normal to the tangent.

We now assume that the acceleration of gravity acts in the direction of the x-coordinate, i.e. perpendicular to the limiting surfaces x = constant. Its magnitude will be ge^{2x} with a constant g, since it is proportional to the number of lines of force piercing a square unit of a limiting surface. A consideration of the potential energy [4, p. 131]

would lead to the same expression. Thus we are ready to set up our equations of motion, based on (9):

$$\ddot{x} + e^{-2x}(\dot{y}^2 + \dot{z}^2) = ge^{2x},$$

$$e^{-x}(\ddot{y} - 2\dot{x}\dot{y}) = 0,$$

$$e^{-x}(\ddot{z} - 2\dot{x}\dot{z}) = 0.$$

It is readily seen, that on account of the last two equations, $e^{-2x}\dot{y}$ and $e^{-2x}\dot{z}$ are equal to arbitrary constants. Hence y and z satisfy a linear equation, which represents a plane cutting the horosphere $x = 0$ orthogonally. We may assume, without loss of generality, that this is the plane $z = 0$, in which x and y are limiting curve coordinates [8, p. 165]. The remaining differential equations are

$$\ddot{x} + e^{-2x}\dot{y}^2 = ge^{2x}, \qquad e^{-x}(\ddot{y} - 2\dot{x}\dot{y}) = 0.$$

For a more detailed classification of the solutions we are going to have, see [2, p. 142], [4, p. 91], and [7, p. 258]. It might also be useful to change the resulting equations back to Weierstrass' coordinates, using (2). We first find $e^{-2x}\dot{y} = C_1$, C_1 being an arbitrary constant. If $C_1 = 0$, the path is a straight line; we now assume $C_1 \neq 0$. If we substitute in our first equation, we obtain by integration

$$\dot{x}^2 = (g - C_1^2)e^{2x} + C_2$$

with a second arbitrary constant C_2. If $g - C_1^2 = 0$ and $C_2 = 0$, we have limiting curves. If $g - C_1^2 = 0$ and $C_2 \neq 0$, we find eliminating t and integrating

$$y = \pm \frac{C_1}{2\sqrt{C_2}} e^{2x} + C_3,$$

where C_3 is another arbitrary constant. Such an equation represents an osculating parabola. Assuming that $g - C_1^2 \neq 0$, integration will yield

$$(y - C_3)^2 = \frac{C_1^2}{g - C_1^2} e^{2x} + \frac{C_1^2 C_2}{(g - C_1^2)^2}.$$

We can distinguish between the following cases:

$g - C_1^2 > 0$,	$C_2 > 0$,	hyperbolic parabola, two branches,
$g - C_1^2 < 0$,	$C_2 > 0$,	hyperbolic parabola, one branch,
$g - C_1^2 > 0$,	$C_2 < 0$,	elliptic parabola,
$g - C_1^2 > 0$,	$C_2 = 0$,	equidistant curve, two branches.

REFERENCES

1. L. Bianchi—M. Lukat, Vorlesungen über Differentialgeometrie, Leipzig, 1899.
2. J. L. Coolidge, The Elements of Non-Euclidean Geometry, London, 1909.
3. L. P. Eisenhart, An Introduction to Differential Geometry, Princeton, 1947.
4. H. Liebmann, Nichteuklidische Geometrie, Dritte Auflage, Berlin and Leipzig, 1923.
5. H. P. Manning, Non-Euclidean Geometry, Boston, 1901.
6. A. D. Michal, Matrix and Tensor Calculus, New York, 1947.
7. D. M. Y. Sommerville, The Elements of Non-Euclidean Geometry, London, 1914.
8. H. E. Wolfe, Introduction to Non-Euclidean Geometry, New York, 1945.

University of California, Davis

THE PERSONAL SIDE OF MATHEMATICS

This department desires especially articles showing what mathematics means to people in various professions and historical articles showing what classic mathematics meant to those who developed it. Material intended for this Department should be sent to the Mathematics Magazine, 14068 Van Nuys Blvd., Pacoima, California.

MATHEMATICS AND THE SPACE-TIME PROBLEM

Roger Osborn

Herein is chronicled a triumph of that often vilified creature, the pure mathematician. This mathematician made a discovery which changed the world's course. One might argue that it would have been changed anyway, but the fact remains that it was changed because of this discovery. This world shaking discovery was the development of non-Euclidean geometry just over one hundred years ago. Much has been written about this discovery and development in its relation to mathematics. Here, then, is an account of its effect on space and time philosophies, the changes in which have brought about the evolution of the theory of relativity.

In order to see how this discovery affected the ideas of space and time it is necessary to review the development of space and time philosophies. It appears that there can be three broad divisions made into which almost all philosophies of the nature of space and time will fall. Some individual philosophies overlap two of these divisions. Taken in the orcer in which they occurred historically, the divisions are: (a) God, (b) Absolute, and (c) Relative. Early philosophers concerned with the problem of space and time identified them with God Himself. Later came the concept that absolute space and absolute time exist independent of any experience of space or time and can not be experienced in themselves. Lastly, came the relative philosophies. These in turn can be divided roughly into three divisions which might be called theories of (1) conceptual, (2) perceptual, and (3) physical space and time. The first two of these are subjective, the first more so than the second, if it can be said that there are degrees of subjectivity. Conceptual space, for example, can be any space (whatever space may be) which the individual may conceive. All mathematical spaces are conceptual. Here is the first inkling of the effect that the development of non-Euclidean geometry was to have. Perceptual space is that space which the individual uses as a frame of reference when perceiving the external world. Physical space is perceptual in nature, but it should have features which are the same for all individual observers. It is listed among the relative spaces since even it is relative to the

position of the observer. Similar explanations might be given for conceptual time, perceptual time, and physical time.

These divisions of the philosophies of space and time lead up to modern times. During this century philosophy has taken a new turn, occasionally suggested before. It seems in the light of modern thought that we live in a world immersed in a space-time system rather than in a system of space and time. The philosophy of a space-time system is basic to the theory of relativity, and hence it could be said to be a relative philosophy, but not in the earlier sense.

All of the above classifications of philosophies of space and time deal with the nature of space and time. Another classification could be set up in which philosophies would be classified according to the manner in which they claim the ideas of space and time are derived. Some claim these ideas are innate. This type of philosophy is no longer held in high esteem. Others claim these ideas are given *a priori*, but are not inborn - they come simultaneous with the first experiences. Still others claim they are abstractions from the ideas obtained from sense-experience. Finally, there are those philosophies which claim that the ideas of space and time are abstractions from a more complex system which is presented to the mind in some manner. All philosophies, except those claiming the ideas of space and time are given innately or *a priori*, include some feature which implies that the mind itself must supply some motive force in obtaining these ideas. (This may be noticed in the philosophy of Sir Arthur Eddington mentioned later in the paper). In this sense, at least, all of these philosophies of space and time or of space-time are subjective.

Still another division of types of philosophies of space and time can be had by considering the composition of space and time. Some philosophies consider them as being composed of an infinite number of infinitesimally small points (from which extension is derived) and instants (from which time is derived). Others take the whole of space and time to be given and obtain parts of this whole only by abstraction. Still others take a middle position, claiming that the basic elements of space and time are chunks of extension and duration. All divisions of these chunks of durations are abstractions, and the totality of space and time, whether finite or infinite, is obtained by the accretion of these appreciable chunks or durations.

An insight may be had into the great change in space-time philosophies if there is an understanding of the philosophies which preceded and followed the discovery of non-Euclidean geometry. This may be accomplished by reviewing the philosophies of various individuals. The following paragraphs will be devoted to a group of fairly brief summaries of some of the views of various philosophers and physical scientists on the subjects of space and time, their meanings, and the means by which we apprehend or perceive them. These philosophies can be classified generally in one or more of the ways given above. No

claim is made for completeness in the summary of any one person's views, nor is it claimed that a valid cross section of all writers has been made. Some authors spend more time and energy in explaining their views of space than of time and vice versa. Included in the summaries are impressions of this writer as well as those of other commentators. It is not the purpose of this section to give arguments for or against any of the views here summarized.

The first modern to discuss the nature of space and time was Nicholas de Cusa (1401-1464). He held that they are products of the mind and hence are inferior in reality to the mind creating them. [VI, p. 59][1] Isaac Barrow (1630-1677) was Sir Isaac Newton's teacher. For him space is a limitless immovable substratum of the universe and time is a capacity or possibility of permanent existence. [XII. p. 50] His was the first clear exposition of the doctrine of absolute space and time.

Sir Isaac Newton (1642-1727) fostered the belief that space and time are, by the will of God, existent in and by themselves, independent alike of the mind which apprehends them, and of the objects with which they are occupied. [I, p. 211] In addition to being independent of the percipient mind, they do not even come under the observation of our senses. [XII, p. 53] He did not define space and time, taking them as being well known to all. [I, p. 182] He did say, though, that, "by existing always and everywhere, God constitutes duration and space." [IX, p. 505]

Gottfried Wilhelm Leibnitz (1646-1716) believed space and time exist only relative to objects and not in their own right. Space is the arrangement of things that co-exist and time is the arrangement of things that succeed one another. Space and time are conceptual or perceptual, but physical space has no real existence. [VI, p. 59] They are abstracted from our confused sense-perceptions of the relations of real things. [I, p. 211]

Immanuel Kant (1724-1804) believed that space has no real existence of its own but is supplied by the mind as a framework for the arrangement of objects. Also, time has no real existence. Whereas space serves for the representation of external perceptions, time serves for the representation of internal perceptions. [VI, p. 59] Even though time is not itself real, the *consciousness* of time, in our apprehension of change, is real, and the same is true for space. [I, p. 211] This does not imply that the ideas of space and time are inborn. They are products of the mind (but not abstractions) given in accordance with unchanging laws by which the mind coordinates sensations. [XII, p. 72] That which does not conform to the forms of space and time cannot be experienced. [XII, pp. 73-74]

At this stage of development there finally appeared the discovery of non-Euclidean geometry with its manifold effects. The works of Lobachevski and Bolyai appeared almost simultaneously, the former in

[1] Roman numerals refer to the bibliography.

1829 and the latter in 1832. Through them mathematicians and philosophers were able to realize that space was not necessarily Euclidean. They found that there were mathematical spaces and physical spaces, and that their properties need not coincide. Their eyes were opened to the fact that any space which could be described mathematically might contain a space concept which could better describe the physical world. [X] It was this break which finally led physicists and philosophers to see that certain physical phenomena might be able to be explained on the basis of some new conception of space which were not able to be explained on the basis of the old. Thus was born the theory of relativity. The following paragraphs outline the philosophies of some of the philosophers, psychologists, and scientists who have developed the philosophy of relative space and time.

William James (1842-1910) as a psychologist, had more to say relative to our modes of becoming aware of time than many philosophers. He contended that the unit of composition of our perception of time is a duration block which is perceived as a whole. Time grows by the accumulation of these finite pieces of time - these duration blocks. [VIII, p. 281]

Alfred North Whitehead (1861-1947) wrote so extensively on his philosophy of space and time, that a brief summary of his position seems almost to be impossible. Whitehead held the view that space and time are not real in themselves, but that we abstract ideas of space and time from events which possess ultimate reality. These events are not atomic. [VII, p. 2] He said that we are not directly aware of "points" and "instants", but that these are abstractions from the general relation of extension among events of which we are directly and empirically aware. [VII, pp. 4-5] We cannot obtain an idea of infinite, unchangeable space from direct observation. [XIV, p. 192] The primary, most concrete element of space is the volume, and that of time is the duration. [XV, p. 95] The idea of space is to be found in the relation of events discernible now, and that of time in the relation of other events to those discernible now. [XIII, p. 53] Even though space and time are abstractions from space-time, time is not space-like and vice versa. [XI, p. 219]

Hermann Minkowski (1864-1909) was a Polish mathematician, who, in 1908, stated the whole content of the theory of relativity in a new and elegant form. Previously, it had been thought that the laws of nature described physical phenomena which occurred in a three-dimensional space, while time flowed uniformly on in another and different dimension. Minkowski proposed that this extra (fourth) dimension is not independent of the three dimensions of space. He introduced a new four-dimensional space in which the ordinarily conceived space contributes three dimensions and time one. This may be called space-time. Every point of space-time is immersed in three dimensions of ordinary space and one dimension of time, and so represents the position of a particle in ordinary space at a particular instant of time. The succession of

positions which a particle occupies in ordinary space at a succession of instants of time is represented by a line in space-time. This he called the world-line of the particle. [V, p. 295]

Sir James Jeans (1877-) is a firm believer in relativity. Space-time is a unity, for him, in which space and time are not entirely separable. This is true since motion through space takes time. Motion cannot be described in terms of a three-dimensional space alone; a fourth dimension must be added. In this space-time unity of four dimensions, it is not possible to determine uniquely which axis is time-like and which three are space-like. The unity is different from its components. Modern physical theory suggests, without being able to prove, that physical space and physical time do not have separate existences. They seem more likely to be abstractions from something more complex—a blend of both. [VI, p. 63]

Albert Einstein (1879-) has, in general, left to philosophers and psychologists the answering of the question of how we perceive space and time. He asserts that space and time have real physical significance, and are not merely fictitious. [II, p. 31] It seems that the experiences of individuals are arranged in a series of events; in this series single events which are remembered appear to be ordered according to the criterion of *earlier* and *later*, which cannot be analyzed further. There exists, then, for each individual, subjective time which is not in itself measurable. By use of speech different individuals can, to some extent, compare these experiences. In this way we find that some sense experiences of different individuals correspond to a certain extent, while some have no such correspondence. We regard as real those perceptions which are common to more than one individual, and which, as a consequence, are more or less impersonal. The conception of physical bodies, especially rigid ones, is a relatively constant complex of such perceptions. The only justification we have for our system of concepts is that they serve to represent the complex of our experiences; they have no legitimacy other than this. It is essential in our judgements and concepts of space to be well aware of the relation of experience to our concepts. For a concept of space, we seem to need the following. New bodies can be formed by bringing bodies B, C, ... up to body A; body A is said to be continued. Body A may be continued in such a way that it comes in contact with any other body X. The ensemble of all continuations of body A is designated as the space of body A. Then it follows that all bodies are in the space of body A (which was arbitrarily chosen). Hence, space in the abstract may not be spoken of, but only the space of body A. [II, pp. 2-3]

Sir Arthur Eddington (1882-) has written extensively for both popular and technical consumption. Of space he has said that there is nothing in our primitive sense experiences which can be designated as spatial, but rather that spatiality seems to be an order of the material objects sensed. The concept of *material object* is logically primary to any concept of space. Material bodies themselves exist only in so far as they are thought. [III, p. 105] Space and time in the

physical world seem to have a togetherness, and neither has an individual existence in the physical world. There is a difference, though, in our apprehension of time-extension and space-extension. All our knowledge of space relations is indirect. It is a matter of inference and interpretation of sense impressions. We have a similar indirect knowledge of time relations, but we also have direct knowledge of them through mentally feeling time. This he calls time of conciousness. This feeling of time is one of the respects in which time differs from space. This time of conciousness may be extended to subjective physical time in the following manner. Sense experiences form a time series indicated by *earlier* and *later*. These series can be repeated by the memory, and they can be repeated with some elements replaced by others by an act of the mind. In this way the time concept is formed as a frame in which experiences may be filled in in various ways. The only reason this cannot become objective is that corresponding series of external events do not appear the same to all individuals. [III, p. 107] Eddington has excited argument over his explanation of how the individual knows that the time of conciousness or time series is uni-directional. He claims that the conciousness can grasp this singleness of direction, but that there is a criterion in the physical world, independent of conciousness, and hence objectively real, establishing this unique direction, and this is the increase in entropy. [I, p. 480]

Thus we have traced the evolution of the philosophy of space and time. No one other single discovery played as important a role in the latest phases of the evolution as did the development of non-Euclidean geometry. That this evolution is surely not yet complete is a view held by many, including Dr. Einstein, whose view is given by his biographer, Philipp Frank: "among the theories there will some day be one which in its logical simplicity as well as its simple representation of observation will be so greatly superior to all rival theories that everyone will recognize it as the best in every respect." [IV, p. 283] Those who developed non-Euclidean geometry have opened areas of thought never before conceived, and it seems sure that some avenue of thought thus opened may lead to even better understanding of the nature of space and time, a thought well expressed by Sir Edmund Whittaker: "The humblest research student was thrilled to feel that the novel and unprecedented types of geometrical form he invented might prove to be not the arbitrary and fanciful creations of a pure mathematician, but a description of the actual universe in which we live." [XVI, p. 41]

BIBLIOGRAPHY

I. Dampier, Sir William C., *A History of Science*, Third edition, New York, The Macmillan Company, 1944.

II. Einstein, Albert, *The Meaning of Relativity*, Princeton, Princeton University Press, 1923.

III. *Encyclopaedia Britannica*, Volume 21, Chicago, Encyclopaedia Britannica, Inc., 1944.

IV. Frank, Philipp, *Einstein: His Life and Times*, New York, Alfred A. Knopf, 1947.

V. Jeans, Sir James, *The Growth of Physical Science*, New York, The Macmillan Company, 1948.

VI. Jeans, Sir James, *Physics and Philosophy*, New York, The Macmillan Company, 1944.

VII. Miller, D. L. and Gentry, G. V., *The Philosophy of A. N. Whitehead*, Minneapolis, Minn., Burgess Publishing Company, 1938.

VIII. *The New International Encyclopaedia*, Second edition, Volume XXII, New York, Dodd, Mead and Company, 1916.

IX. Newton, Isaac, *The Mathematical Principles of Natural Philosophy*, translated by Andrew Motte, New York, D. Adee, 1848.

X. Osborn, Roger, "Some Historic and Philosophic Aspects of Geometry," *Mathematics Magazine*, Pacoima, Calif., Mathematics Magazine, Nov.-Dec., 1950.

XI. Schilpp, Paul A., editor, *The Philosophy of Alfred North Whitehead*, Evanston, Ill., Northwestern University Press, 1941.

XII. Werkmeister, W. H., *A Philosophy of Science*, New York, Harper and Brothers, Publishers, 1940.

XIII. Whitehead, A. N., *The Concept of Nature*, Cambridge, Cambridge University press, 1930.

XIV. Whitehead, A. N., *The Organization of Thought*, Philadelphia, J. B. Lippincott Company, 1917.

XV. Whitehead, A. N., *Science and the Modern World*, New York, The Macmillan Company, 1939.

XVI. Whittaker, Sir Edmund, "Mathematics," *Scientific American*, Vol. 183, No. 3, New York, Scientific American, Inc., Sept., 1950.

University of Texas

We think that our organization could be of great interest for your readers and subscribers, as we are at their disposal to give them free information on all books and periodicals published not only in France, but also in Europe in all fields. We can also send them all catalogues that might interest them and provide them all European books and periodicals at French retail prices, the only extra charges being the mail expenses (all our packages being sent by registered mail). We are also in position to secure for them rare and out of print books or to send them micro-films of these books.

STATISTICAL QUALITY CONTROL

John M. Howell

Statistical Quality Control is the application of statistical methods to manufacturing and industrial problems. These methods, which were first employed less than thirty years ago, are widely used today.

The first assembly operation was performed many thousands of years ago when some stone age man fitted a stone to a wooden handle by means of some crude rope to make the first axe. Manufacturing in ancient times and up until less than two centuries ago was on a custom basis. One could not drive in at the corner service station to get a new wheel if he broke one on his wagon. A new one would have to be made and fitted in place. One of the biggest improvements in manufacturing practice was made just before 1800, when Eli Whitney made some parts for guns which were interchangeable. It was the common thing during this period for one man to make his own tools, make the parts, inspect them, and then assemble them. Today, workmen usually perform very specialized functions. One of these specialized functions in a manufacturing plant is Quality Control.

The first use of statistical methods to improve a manufacturing process was in the early 1920's. This was done by Dr. Walter Shewhart at the Bell Telephone Company. Since that time, the use of these techniques has spread to nearly all of the large manufacturing companies and many of the small ones. This growth was rather slow at first, but due primarily to the stimulus of the last war, these methods have become very widely used in recent years. Several companies report savings in the millions due to use of quality control techniques.

Statistics is a field which presents many opportunities. Quality Control presents splendid opportunities for young men who have some knowledge of statistics and also some knowledge of engineering or manufacturing processes. There are also tremendous possibilities for the application of these techniques to fields other than the manufacture of machine parts. Quality Control has been applied to chemical processes, packaging of food and other materials. Here is an answer for a mathematics teacher who may be asked the question, "What can I do with mathematics other than teach?"

The field of Quality Control can be divided into three branches: process control, acceptance sampling, and research and development. A brief description of some of the methods used will be given here.

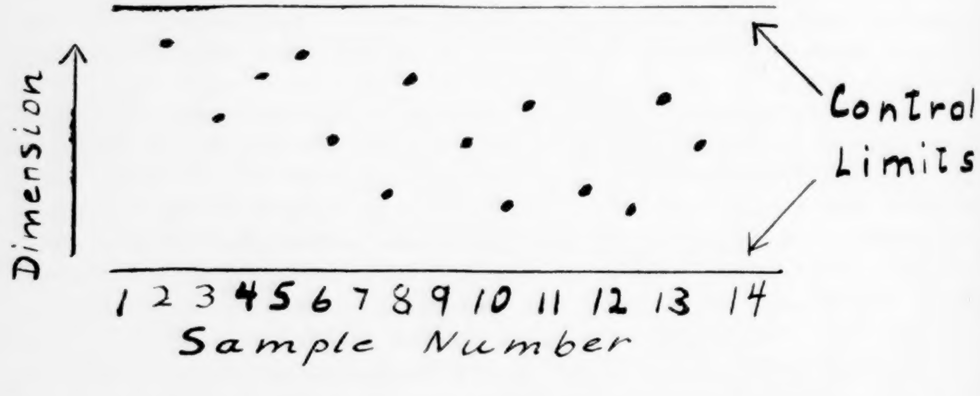
Process control concerns a manufacturer's inspection of his own product in order to determine:

1. what to do with the parts which have been made, (accept, rework, or reject)
2. what to do about the manufacturing process, (change it or not).

This control is usually accomplished by statistical tools called control

charts.

If measurements are made, charts for averages and ranges are used. These measurements might be dimensions, electrical characteristics, or weights. A small sample, often five pieces, is inspected at approximately regular intervals. This may include all of the product or only a portion of it. The average and the range of the sample are computed, the range being the difference between the smallest and largest measurement. The value of average is plotted on one chart and the value of range on another. After about twenty such points have been plotted on the charts, control limits are computed. This calculation requires merely that the average values found be multiplied by some constants which are found in any text on the subject. These limits are placed on the charts as horizontal lines. To simplify the discussion here, we will assume that all points for this preliminary data are within these control limits, or we say the process is "in control". A typical control chart which is in control then appears as follows:



As the manufacturing process continues, points are plotted on the control chart and if the points remain within the control limits, this fact is taken as evidence of control. But if a point should fall outside of control limits, conditions at this time are noted and attempts made to bring the process back into control. This takes only a little statistical knowledge, but a large amount of practical knowledge. The usual thing is to find a process "out of control" at the beginning, but discussion of this case is considerably more complex and will not be included here.

If measurements are not made on the product, but it is simply divided into two portions: that which is good and that which is not good, the percent defective of each sample is plotted on a chart. Limits for this type of chart may also be found and the results are very similar to that given above for the case where measurements are made. For this type of chart, called a percent defective or fraction defective chart,

larger sample sizes are necessary and results are often not as positive. This type of chart is often used for sub-assemblies and in the inspection of protective or decorative finishes where measurements are impossible or impracticable.

For large assemblies, a chart known as a defects per unit chart is often made. This chart has the number of defects plotted in place of measurements. Further information on process control may be obtained from Grant, "Statistical Quality Control".

Acceptance sampling is used in the inspection of incoming material. Here the purpose of inspection is to determine:

1. what to do with the particular lot at hand, (accept, rework, or reject)
2. what to do about the supplier, (continue to purchase from him or not).

A statistical sampling plan indicates how much inspection is required for any desired value of percent defective in the material. The first sampling plans were devised by Dodge and Romig of the Bell Telephone Company in the 1920's. A new sampling plan for the procurement of military material, MIL 105A, has appeared within the last year and unifies procedures for the various branches of the armed services. By using statistical sampling plans, a manufacturer can tell if the material and component parts he is getting are good enough for his needs.

Sampling by variable, that is by using measurements instead of merely determining conformance, is a new field, and although some plans of this type are being developed, none has been released for general use up to the present time. Those interested in the subject of acceptance sampling should read, "Sampling Inspection", by Freeman, Friedman, Mosteller, and Wallis.

The use of statistical techniques in research and development is a vast and interesting field. Here, a much broader background in mathematics and statistics is necessary to cope with the problems which arise. The reader is referred here to Freeman, "Industrial Statistics".

Other references on this subject can be found in those cited and in Butterbaugh, "A Bibliography of Quality Control", or in "Industrial Quality Control", which is the publication of the American Society for Quality Control.

Los Angeles City College

MATHEMATICAL CAREERS IN MILITARY RESEARCH

John W. Odle

Mathematicians who can recall conditions prior to World War II are aware that tremendous changes have occurred in the opportunities open to them. Formerly, teaching was almost the sole occupation available, with a scarcity of openings and anemic salaries. Now the mathematician is in the enviable position of being able to select from a wide variety of well-paid jobs in government and industry, as well as in the field of education. The excellent work done by the Applied Mathematics Panel and by many scattered individual mathematicians during the war helped greatly to bring about a recognition of the usefulness of mathematicians in activities concerned with the development and usage of complicated equipment. This recognition did not die out after the war and, in fact, is still on the increase.

In this presentation primary attention will be focused on those outlets for mathematical talent which are directly connected with military agencies. As all taxpayers well know, the defense establishment in this country is heavily engaged in the support of many weapon development projects, in both government and industrial laboratories. Modern weaponeering has become a tremendously complicated business requiring the best possible talent in the fields of physics, chemistry, and engineering. Because mathematics is the universal tool in these fields, and because the workers in these fields are not all mathematical wizards, it is natural and inevitable that mathematicians should be drawn in to participate in the formulation and solution of problems.

As one would naturally expect, the primary demand is for mathematicians with special training and experience in some applied field, although a surprising amount of pure mathematical research is sponsored by the Office of Naval Research and certain Air Force and AEC agencies. Security restrictions prevent detailed discussion of many of the interesting problems, but certainly it can give little aid or comfort to the enemy to know that our scientists are pursuing studies in such fields as aerodynamics, ballistics, fluid mechanics, circuit analysis, thermodynamics, elasticity, theoretical mechanics, optics, nucleonics, and other branches of mathematical physics. In addition to their participation in such studies, mathematicians are contributing important services of general utility in the areas of high-speed computing and statistical analysis.

Computing, as a matter of fact, has become one of the major new fields in mathematics. It is, of course, the oldest branch of mathematics, but the accomplishments of modern electronics have excited many new interests in treating by numerical methods problems which are analytically intractable. The government is subsidizing many super-computer development projects, and almost every laboratory has plans, or at least hopes, for

acquiring one of the new monsters. Fortunately, there seems to be no danger of technological unemployment from these developments. On the contrary, any institution acquiring modern computing equipment always seems to develop a voracious appetite for more mathematicians, because the machinery has not yet acquired independent volition and someone has to tell it in precise detail just what to do. Modern computing is an excellent field to recommend to budding young candidates for a career in mathematics because it offers opportunities for people with a very wide range of training - from a taste of undergraduate mathematics to the Ph.D. level - and the market should be an expanding one for a long time to come.

A new field for mathematicians, and other scientifically trained people, which opened up during the past war is that of operations research. The armed services found that the efficiency of operations could sometimes be dramatically improved by relatively simple changes in procedure recommended by scientific observers, and a whole new service was born to supply such observers. The Navy now has an Operations Evaluation Group, the Army an Operations Research Office, and the Air Force an Operations Analysis Section. These groups, despite the minor variations in name, all have essentially the same purpose, namely, to subject military doctrine, tactics, and weapon planning to critical scientific scrutiny and suggest improvements to various command levels. The techniques of statistics and probability are the chief mathematical tools for such work. For mathematicians with a high sense of adventure, military operations research offers an exciting and rewarding career. As a bit of advice gleaned from personal experience, it might be mentioned that this work is particularly suitable for single men because it usually involves a considerable amount of travel about the world. Wives tend toward a jaundiced view of this feature of the work. The field is not generally open to women because of the complications of arranging living accommodations when with the armed forces away from home.

As an indication of the extent to which mathematicians are being used in the defense establishment, the writer recently jotted down for his own use a casual list of the military offices and laboratories which he knew by personal experience hired high level mathematicians, and with no difficulty ran the number up beyond thirty. This was by no means a complete list, and it did not include the many university and industrial groups doing military research and development work on contracts.

For a mathematics student considering an eventual career in military research and development there are two sound recommendations. One is to get the maximum amount of education, up to and including the Ph.D. degree, if possible, and the other is to develop a strong interest in some applied field. A minor in physics is very desirable. Mathematical statistics is also an excellent field to concentrate in at this time. It is assumed, of course, that anyone majoring in mathematics will automatically get a thorough grounding in the fundamentals of analysis. Without such a foundation, one can become at best only a handbook

specialist with no capacity for advanced research.

A point of view which needs to be heavily stressed in the training of mathematicians for non-academic careers is that problems are not considered solved until usable numerical solutions are obtained. The consumer is more interested in results than in the ingenuity which may have been required to get them. Furthermore, the readers for whom reports on work accomplished are intended are often not trained in the intricacies of mathematical analysis, and consequently the essential results must be presented with great clarity and simplicity.

Civilian employees of the military services come under the provisions of the Federal Civil Service regulations, with all of the advantages and disadvantages which Congress has chosen to impose. Actual working conditions vary widely, of course, from one laboratory or office to another, but there are certain constants throughout the system. For example, annual leave for vacations is granted according to a definite formula: Thirteen working days per year for employees with less than three years federal service, civilian and military; twenty working days for employees with three to fifteen years service; and twenty-six working days for employees with more than fifteen years service. In addition, thirteen working days of sick leave are allowed per year. These leave credits may be accumulated from year to year subject to a maximum limit of sixty days annual leave time and with no limit on sick leave. Pay scales are supposed to be uniform also, with salaries determined according to an objective system for rating the difficulty and importance of each position. Eighteen different levels are recognized under the present government schedule for per annum employees, labelled GS-1 to GS-18. Each level has a basic salary and a schedule of within-grade increases.

To give some concrete examples, the GS-5 level, which is the normal entering grade for a fresh college graduate with no other experience, has a starting salary of $3410, with annual increases of $125 up to a maximum of $4160. A Master's degree qualifies a candidate for a grade of GS-7, with a starting salary of $4205 and annual increases of $125 up to a maximum of $4955. A Ph.D. degree now is sufficient qualification for a GS-11 position, with an entering salary of $5940 and a $200 increase each 18 months up to a maximum of $6940. Promotions beyond these levels are dependent upon individual initiative and the availability of openings. Base salaries at the higher levels are as follows: GS-12, $7040; GS-13, $8360; GS-14, $9600; GS-15, $10,800. Appointments to grades GS-16 to GS-18, salaries $12,000 to $14,800, require special approval and are limited in number by statute.

Civil Service employees also enjoy an excellent retirement system and good job security. The latter item is determined chiefly by Congressional appropriations. Anyone is free to make his own predictions, but it appears highly probable that the military services will continue to receive strong support for a long time to come. Furthermore, scientific staffs are among the last to be let go at times of cutting back.

With regard to the less tangible but nevertheless vitally important factor of job satisfaction, this obviously depends on the individual and on the particular local environment he finds himself in. One can find both good and bad situations in government service, just as in any other field of employment. On the whole, the opportunities for doing satisfying, high-level scientific work are now as good in military laboratories as in any other area with which the writer is familiar.

On the debit side of the ledger one can cite a reasonably impressive list of disadvantages also. One is the ever-present red tape which is the source of many jokes and much frustration. However, red tape is apparently no longer a government monopoly, because one hears just about as much complaining on this score from academicians as from civil servants. Actually, red tape seems to be an inevitable accompaniment of bigness in any enterprise, whether public or private. Examples of problems which bigness creates are: delays in getting decisions made, communication difficulties in dealing with unknown bosses far away, submergence of the individual and a lessening of his feeling of effectiveness, necessity for voluminous progress reports, and difficulties in getting new projects underway. Of course, in fairness it should be pointed out that bigness also makes possible the expenditure of large sums and the undertaking of large-scale projects impossible in a small organization.

A further disadvantage connected with doing technical and scientific work for military agencies is that much of the work is classified, and hence there are restrictions on publishing or disclosing accomplishments. This appears to be a necessary restraint which can never be eliminated.

As in almost any choice one may make in life, one has to take the bitter with the sweet, and government service is no exception. Fortunately, in the minds of enough people, the advantages seem to outweigh the disadvantages so that necessary jobs manage to get done. There is still room for further talent, however.

U. S. Naval Ordnance Test Station
China Lake, California

MISCELLANEOUS NOTES

Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

SOME NOTES ON THE LIMIT CONCEPT

The intriguing relationship between the operation leading to the basic geometrical interpretation of the derivative and that which brings forth the function e^x seem to be ignored in most calculus texts. An understanding of this relationship not only strengthens the concept of limit acquired by a calculus student - but it helps to remove the mystery associated with e^x.

Without further ado let us recall that the elementary geometrical illustration of the derivative entails a single-valued continuous curve $f(x)$ which is differentiable at all points in the given interval.

A secant line may then be constructed through a fixed point P and any other point, say Q, on the curve. The usual procedure is to permit the point Q to approach the point P along $f(x)$. As Q approaches P, the slope of the secant line approaches the slope of the tangent line at the point P. The limit of the slope of the secant line *IS* identical with the slope of the tangent. In this operation the curve $f(x)$ is held stationary while the straight line through P is rotated. It will be shown that the exact opposite is true with the function e^x, i.e. the straight line is held constant and the curve $f(x)$ rotated!

Consider the points of intersection between the line (a) $y = x + 1$ and a curve of the form (b) $y = a^x$. It is evident that the curves will meet at $x = 0$, independent of the value of "a". Call this the fixed point P. Now to find the other junction, eliminate y between equation (a) and (b). Thus $a^x = (x + 1)$ or

Eq. (c) $$a = (x + 1)^{1/x}$$

Equation (c) represents the value of "a" for a given x such that (a) and (b) will meet at a point, say point Q. If we let Q approach the point P along the straight line (a), the curve $f(x) = a^x$ will experience a rotation about P. The limiting position of this rotation, as in the previous case, is unique in that it will touch the straight line at only one point. The value of "a" in the limit is "e". that is:

$$\lim_{x \to 0} a = \lim_{x \to 0} (1 + x)^{1/x} = e$$

and the curve of form (b) becomes $y = e^x$.

Figure (1) is the typical illustration of the derivative for $y = x^2$, while figure (2) represents the foregoing process.

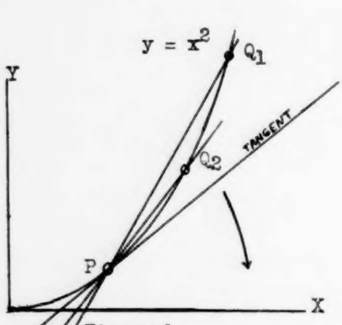


Figure 1.

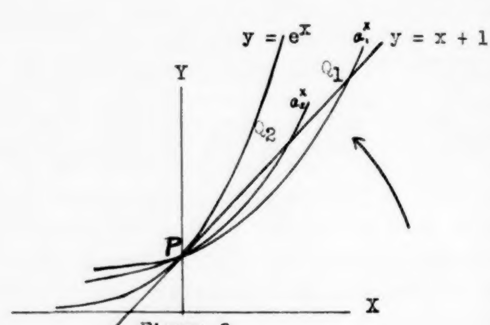


Figure 2.

The above examples illustrate the most important characteristic of every *actual* ((as opposed to a relative lim)) limit process. That is, a limit must always be outside the domain of that which it limits. For instance, no member of the subclass secant, in the first example, may be called a limit, for another can always be found whose slope exceeds the one mentioned.

A simple illustration of this is found in the class of polygons inscribed in a circle. As the number of sides of the polygons become as large as we desire, the area of the polygon approaches the area of the circle. However, none of the elements (polygons) of the given sub-class may be called an actual limit, for if it were, another polygon could be found whose sides are two, three, etc. times as numerous as the sides of the given one; and consequently, the area of the larger polygon would more closely approximate the area of the circle. The areas of the polygons are forever seeking but never attaining the area of the circle, which is obviously the limit.

It is interesting to note that a whole field of mathematics arose by predicating the property of one subclass with that of another under the same genus. Consider the genus Magnitude, if the property of Point is predicated of the subclass Line, then a "quantity" is produced which is so small that it can't get any smaller. This is the so called infinitesimal which, thanks to Weierstrass, is being recognized by mathematicians for what it is - a useless metaphor!

Leibniz utilized this concept when he predicated secant of the tangent. He said, in effect, that a tangent touches a curve at two points which are infinitesimally close. Poisson[1], as well as Leibniz, believed that infinitesimals actually existed.

Mathematicians were not the only ones who misused interclass pred-

[1] **Poisson, Traité de Mécanique, Part I, second edition, p. 14. Paris, 1833.**

ication. The world is fully cognizant of the opportune employment of these infinitely small quantities by Hegel in his philosophical doctrines. The same error is repeated by anthropologists when they predicate brute of man and thereby produce the fictitious "missing link" which has, of necessity, the same kind of reality as the infinitesimal.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

Tables to Facilitate Sequential t-Tests, by Kenneth J. Arnold, National Bureau of Standards Applied Mathematics Series 7, xix, 82 pages, 45 cents (order from Government Printing Office, Washington 25, D.C.).

This 82-page booklet will be of interest and value especially to statisticians and research workers in the physical and biological sciences, in engineering, and in industrial quality control. It will enable them to answer economically the commonly occurring question whether or not a certain specified value is the mean of a normal population with unknown dispersion.

Sequential analysis is a newly developed tool of statistical sampling. Instead of taking a sample of fixed size, the investigator uses size dictated by the outcome of the observations. This usually allows a smaller sample than those under previous methods in common use. The tables will make it possible to apply the efficient methods of sequential to the testing of hypotheses regarding the mean of a normal population.

In the application of this new tool, each type of test requires special tables to determine whether or not the accumulated evidence from the observations at each stage calls for additional data or justifies one decision or another. Using the present tables, the investigator decides the possibility of the mean being a certain given number in the important type of tests in which a normal universe with unknown mean and dispersion is given. The appearance of these tables should considerably further the many practical applications of sequential analysis.

General Homogeneous Coordinates in Space of Three Dimensions. by E. A. Maxwell. Cambridge University Press, New York, 1951. $2.75.

This text deals primarily with classical elementary analytic projective geometry of three dimensions, with especial emphasis on the properties of quadric surfaces, the elements of line geometry, and the elementary properties of twisted cubic curves. No use is made of matric notation, though an indication of its value is given in the final chapter.

The book is a sequel to the author's work on two-dimensional projective geometry, and references are made to this work. Some knowledge of this subject is necessary for an understanding of the text under consideration.

The work is tersely written, which would make it difficult reading for the average American student. There are no suggestive illustrations, few illustrative examples, and few simple exercises. However, many of the exercises deal with interesting properties of curves and surfaces, and some contain theory essential to the understanding of the subject.

It should be noted that this type of geometry is not at present popular in this country. However, this book should prove to be of value as supplementary reading for those interested in this subject.

R. C. Sanger

PROBLEMS AND QUESTIONS

Edited by

C. W. Trigg, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject-matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by such information as will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction. Readers are invited to offer heuristic discussions in addition to formal solutions.

Send all communications for this department to *C. W. Trigg, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

119. *Proposed by P. A. Piza, San Juan, Puerto Rico.*

Solve $t_x t_{x+1} t_{x+2} = 468 t_y^3 + 468$ for x and y. $t_a = a(a + 1)/2$.

120. *Proposed by Victor Thébault, Tennie, Sarthe, France.*

On the sides CB and CD of rectangle $ABCD$ construct internally (or externally) equilateral triangles CEB and DFC. Show that triangle AEF is equilateral.

121. *Proposed by Norman Anning, University of Michigan.*

Solve in positive integers, $(x + iy)^3 = x +$ (a pure imaginary). For instance, $(7 + 4i)^3 = 7 + 524i$.

122. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D.C.*

Show that the envelope of a circle, the square of whose tangent from the origin is equal to the ratio of the abscissa to the ordinate of its center, the center lying on the parabola $x = ay^2$, is a circular cubic, one of whose asymptotes is parallel to the y-axis.

123. *Proposed by Joseph Barnett, Jr., Clarksburg, W. Va.*

Theorem: A necessary and sufficient condition that a perpendicular from the vertex of a spherical triangle to the circle containing the opposite side fall on that side is that the angles adjacent to that side be of the same species.

124. *Proposed by Leo Moser, University of Alberta, Canada.*

Prove that if p and q are integers not exceeding the integer n, then it is possible to arrange n or fewer unit resistances to give a combined resistance of p/q.

125. *Proposed by William Leong, University of California at Berkeley.*

Consider the sequence of numbers $\{a_i\}$ where $3a_1 = 1$, $7a_2 = a_1^2$, $11a_3 = 2a_1a_2$, $15a_4 = a_2^2 + 2a_1a_3$, $19a_5 = 2(a_1a_4 + a_2a_3)$, $23a_6 = a_3^2 + 2(a_1a_5 + a_2a_4)$, $\cdots$. Let $\mu^4 = \lim_{n\to\infty} a_n/a_{n+1}$. Then show that (a) the number μ exists, and (b) μ satisfies the equation

$$\sum_{k=1}^{\infty} \frac{(-1)^k \mu^{4k}}{3\cdot4\cdot7\cdot8\cdots(4k-5)(4k-4)(4k-1)4k} = 0.$$

SOLUTIONS

Erratum

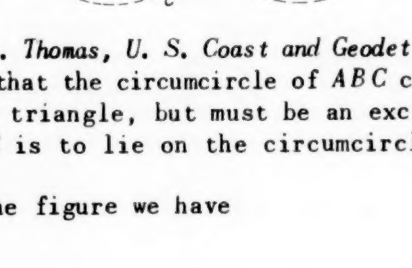
89. In the solution of this problem which appeared on page 109 of the November-December 1951 issue the 23rd line ends with "The desired equations are then ..." This should have been followed in the printed solution by

"t_i: $y = m_i x + b_i$ and u_i: $y = m_i x + c_i$ where"

A Condition for $a^2 + b^2 + c^2 = 6R^2$

83. [November 1950] *Proposed by Victor Thébault, Tennie, Sarthe, France.*

If the Lemoine point of a triangle ABC lies on the circumcircle of the tangential triangle of ABC we have $a^2 + b^2 + c^2 = 6R^2$, where a, b, c are the sides and R is the circumradius of ABC.



Solution by P. D. Thomas, U. S. Coast and Geodetic Survey, Washington, D.C. It is clear that the circumcircle of ABC cannot be the incircle of the tangential triangle, but must be an excircle if the Lemoine point, K, of ABC is to lie on the circumcircle of the tangential triangle.

Referring to the figure we have

$$\overline{OO'}^2 - R'^2 = 2RR', \tag{1}$$

where O, R and O', R' are the circumcenters and circumradii respectively of ABC and its tangential triangle $A'B'C'$.

With respect to the inscribed quadrilateral $A'B'KC'$ we have

$$\overline{BC}^2 = a^2 = \overline{O'B}^2 + \overline{O'C}^2 - 2R'^2. \tag{2}$$

Again from the figure, using the median theorem in triangle $O'BC$, we have

$$2\,\overline{O'S}^2 = \overline{O'B}^2 + \overline{O'C}^2 - \tfrac{1}{2}a^2. \tag{3}$$

Considering the triangle $O'A'O$ we find by Stewart's theorem

$$\overline{O'S}^2 = R'^2 + a^2(\overline{OO'}^2 - R'^2)/4R^2 - a^2/4. \tag{4}$$

From (2) and (3) we have $\overline{O'S}^2 = R'^2 + a^2/4$ and with the value of $\overline{OO'}^2 - R'^2$ from (1) we may write (4) as

$$R'^2 + a^2/4 = R'^2 + a^2R'/2R - a^2/4, \text{ whence } R = R'.$$

With this condition the relations (1) and (2) become

$$\overline{OO'}^2 = 3R^2,\ \overline{O'B}^2 + \overline{O'C}^2 = 2R^2 + a^2. \tag{5}$$

Now if F is the midpoint of $\overline{OO'}$, we find in the quadrilateral $O'BOC$

$$(\overline{O'B}^2 + \overline{O'C}^2) + 2R^2 = \overline{OO'}^2 + a^2 + 4\overline{FS}^2. \tag{6}$$

With the values from (5) placed in (6) find that $\overline{FS} = R/2$.

Similarly it may be shown that the distance from F to the midpoints of AC and AB is equal to $R/2$, that is F is the nine point center of ABC and O' is therefore the orthocenter of ABC. In view of this last fact we have

$$\overline{O'C}^2 + c^2 = 4R^2,\ \overline{O'B}^2 + b^2 = 4R^2 \text{ or}$$

$$(\overline{O'B}^2 + \overline{O'C}^2) + b^2 + c^2 = 8R^2. \tag{7}$$

Substituting from (5) in (7) we have the announced relation.

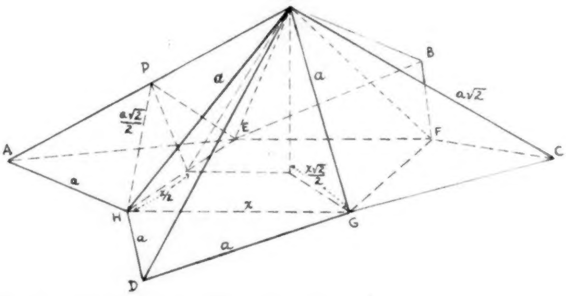
The relations (1) and (2) may be found in Altshiller-Court, *College Geometry*, pages 109, 114. Relations (3), (4), (6) and (7) appear in R. A. Johnson, *Modern Geometry*, pages 68, 163, 191.

Also solved by *L. M. Kelly, Michigan State College*.

Volume of Solid formed by Folding

95. [March 1951] *Proposed by A. O. Qualley, Farnhamville Independent School, Iowa.*

Each side of a square sheet of paper $ABCD$ equals $2a$. The midpoints of AB, BC, CD, and DA are E, F, G, and H, respectively, and O is the center of the square. The square is creased inward along EG and FH, and outward along the diagonals AC and BD. E, F, G, and H are then drawn inward to form the base of a square pyramid $O - EFGH$, surrounded by four tetrahedra $OAHE$, $OBEF$, $OCFG$, and $ODGH$. (1) If each side of the base $EFGH$ is a, find the volume of the whole solid. (2) Find the length of the side of the base $EFGH$ for which the total volume is a maximum, and compute the maximum volume.



Solution by Leon Bankoff, Los Angeles, California. Let x represent the length of the side of the base of the square pyramid $O - EFGH$. Then the volume of the pyramid is $(x^2/3)\sqrt{a^2 - x^2/2}$. A plane through HE and P, the midpoint of AO, is perpendicular to AO and divides the tetrahedron $OAHE$ into two congruent tetrahedra with isosceles triangular bases. The volume of each of these tetrahedra is $(1/3)(a\sqrt{2}/2)(x/2)\sqrt{a^2/2 - x^2/4}$ or $(ax/12)\sqrt{a^2 - x^2/2}$. There are eight of these tetrahedra, so the volume of the whole solid is

$$V = [x^2/3 + 2ax/3]\sqrt{a^2 - x^2/2}. \qquad \text{(I)}$$

(1) Hence, when $x = a$, $V = a^3\sqrt{2}/2$ or approximately $0.7071a^3$.
(2) Obtaining dV/dx from (I), equating it to zero and simplifying, we have

$$3x^3 + 4ax^2 - 4a^2x - 4a^3 = 0.$$

The one positive root of this equation is approximately $1.07170a$. Therefore, from (I), the maximum total volume of the whole solid is approximately $0.7160a^3$.

Also solved by *A. L. Epstein, Cambridge, Mass.; and the proposer.*

Difference of a Cube and a Square

98. [May 1951] *Proposed by Victor Thébault, Tennie, Sarthe, France.*

Find a perfect cube and a perfect square such that their difference is 2,000,000.

Solution by Leon Bankoff, Los Angeles, California. All solutions of the equation $x^2 + 2y^2 = z^n$ in relatively prime x, y and positive z are given by $x + y\sqrt{-2} = \pm(p + q\sqrt{-2})^n$, $z = p^2 + 2q^2$ with p, $2q$ relatively prime and of different parity. [See, e.g., Uspensky and Heaslet, *Elementary Number Theory*, McGraw-Hill, (1939), p. 393-5.]

If $y = 1000$, $n = 3$, then $z^3 - x^2 = 2{,}000{,}000$ and $x + 1000\sqrt{-2} = \pm(p + q\sqrt{-2})^3 = \pm(p^3 + 3p^2q\sqrt{-2} - 6pq^2 - 2q^3\sqrt{-2})$. Equating the real parts and the imaginary parts, we have

$$x = \pm(p^3 - 6pq^2) \text{ and } 3p^2q - 2q^3 = \pm 1000,$$

whereupon $p = \pm\sqrt{[2q^2 \pm 1000/q]/3}$, where the double signs are independent of each other. Hence q must be a divisor of 1000. Now 8 is the only permissible value of q of parity different from the corresponding p, namely ± 1. Therefore $x = \pm 383$, $z = 129$ and

$$129^3 - 383^2 = 2146689 - 146689 = 2000000.$$

Also solved by George Baker, California Institute of Technology; H. H. Berry, University of Kentucky; Monte Dernham, San Francisco, California; L. A. Ringenberg, Eastern Illinois State College; and the proposer.

Baker, by inspecting a table of squares and cubes also found $(300)^3 - (5000)^2 = 2{,}000{,}000$. This result may also be obtained by taking $p = q = 10$ in Bankoff's solution. That this is the only other solution to $z^3 - x^2 = 2{,}000{,}000$ may be shown by assuming that z and x have certain common factors and reducing the resulting equations. The only ones secured are

$$z_1^3 - x_1^2 = 31250, \quad z = 4z_1, \quad x = 8x_1;$$
$$z_2^3 - x_2^2 = 128, \quad z = 25z_2, \quad x = 125x_2; \text{ and}$$
$$z_3^3 - x_3^2 = 2, \quad z = 100z_3, \quad x = 1000x_3.$$

Following the method of Bankoff's solution, we find $x_1 = \pm 625$, $z_1 = 75$; $x_2 = \pm 40$, $z_2 = 12$; $x_3 = \pm 5$, $z_3 = 3$. Each of these solutions is equivalent to $x = \pm 5000$, $z = 300$.

Area of Parallelogram Circumscribed to Ellipse

100. [May 1951] *Proposed by Wang Shik Ming, Chung Hwa High School, Malang, Java, Indonesia.*

The area of the parallelogram formed by the tangents to an ellipse

at the extremities of any pair of conjugate diameters is equal to the area of the rectangle contained by the axes of the ellipse.

Solution by Charles McCracken, Jr., University of Cincinnati. Let C be the center of the ellipse and let PCP', DCD' be the conjugate diameters. The area of the parallelogram which touches the ellipse $x^2/a^2 + y^2/b^2 = 1$ at P, P', D, D' is $4(CD)(CF)$ where CF is the perpendicular from C on the tangent at P.

Now if the eccentric angle of P is ϕ, the eccentric angle of D is $\phi \pm \pi/2$. Then

$$(CD)^2 = a^2\cos^2(\phi \pm \pi/2) + b^2\sin^2(\phi \pm \pi/2)$$

or

$$(CD)^2 = a^2\sin^2\phi + b^2\cos^2\phi. \tag{1}$$

The equation of the tangent at P is

$$(x/a) \cos \phi + (y/b) \sin \phi = 1.$$

So

$$(CF)^2 = 1/[(\cos \phi)/a^2 + (\sin \phi)/b^2]$$

or

$$(CF)^2 = a^2b^2/(a^2\sin^2\phi + b^2\cos^2\phi). \tag{2}$$

From (1) and (2) we see that the area of the parallelogram is equal to $4ab$. Since the major and minor axes are conjugate diameters, the proposition is proven.

The above proof appears as a theorem on pages 132-133 of Charles Smith, *Conic Sections* (1884).

Also solved by *George Baker, Student, California Institute of Technology; Leon Bankoff, Los Angeles, California; Vern Hoggatt and Adrian Wenner, Oregon State College.*

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q 51. Compute the first period of the repeating decimal equivalent to $1/7^2$. [*Submitted by J. M. Howell.*]

Q 52. Prove that no perfect square can be written in the scale of ten with just five digits which are distinct, but congruent modulo 2. [*Victor Thébault in the American Mathematical Monthly*, **44**, 248, (April 1937).]

Q 53. Solve for z: $x(x + 1) + y(y + 1) + z(z + 1) = 5/2$.

Q 54. Prove that the area of a parallelogram, whose vertices lie at lattice points of a square lattice, is a whole number of unit squares. [*Submitted by Leo Moser.*]

ANSWERS

A 51. $1/7^2 = 1/49 = (0.02)/(1 - 0.02) = (0.02) + (0.02)^2 + (0.02)^3 + \cdots$
$= 0.020408163264$

$$\begin{array}{r}
0.020408163264 \\
128 \\
256 \\
512 \\
1024 \\
2048 \\
4096 \\
8192 \\
16384 \\
32768 \\
65536 \\
131072 \\
262144 \\
524288 \\
1048576 \\
2097152 \\
4194304 \\
8388608 \\
16777216 \\
33554432 \\
67108864 \\
\underline{134217728} \\
0.02040816326530612244897959183673469387755102040
\end{array}$$

A 52. There are but two sets of five digits, 0 2 4 6 8 and 1 3 5 7 9, which are distinct and congruent modulo 2. The sum of the digits of every perfect square must be congruent modulo 9 to 0, 1, 4 or 7. However, the sum of the digits of the first set is congruent to 2 (mod 9). If the last digit of a perfect square is odd, the penultimate digit must be even. The second set contains no even digit. Hence no permutation of either set can be a square number.

A 53. $z/(z + 1) = 5/2 - x/(x + 1) - y/(y + 1)$
$= (5xy + 5x + 5y + 5 - 2xy - 2x - 2xy - 2y)/2(x + 1)(y + 1)$
$= (xy + 3x + 3y + 5)/(2xy + 2x + 2y + 2)$. Applying division we have:
$z = (xy + 3x + 3y + 5)/(xy - x - y - 3)$.

A 54. The result follows immediately from the determinant expression for the area of a parallelogram (or triangle) derived in elementary analytic geometry texts. The value of a determinant whose elements are integers is of course an integer.